

THE VELDKAMP SPACE OF THE SMALLEST SLIM DENSE NEAR HEXAGON

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We give a detailed description of the Veldkamp space of the smallest slim dense near hexagon. This space is isomorphic to $PG(7, 2)$ and its $2^8 - 1 = 255$ Veldkamp points (that is, geometric hyperplanes of the near hexagon) fall into five distinct classes, each of which is uniquely characterized by the number of points/lines as well as by a sequence of the cardinalities of points of given orders and/or that of (grid-)quads of given types. For each type we also give its weight, stabilizer group within the full automorphism group of the near hexagon and the total number of copies. The totality of $(255 \text{ choose } 2)/3 = 10,795$ Veldkamp lines split into 41 different types. We give a complete classification of them in terms of the properties of their cores (i.e. subconfigurations of points and lines common to all the three hyperplanes comprising a given Veldkamp line) and the types of the hyperplanes they are composed of. These findings may lend themselves into important physical applications, especially in view of recent emergence of a variety of closely related finite geometrical concepts linking quantum information with black holes.

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1. Introduction

It has only relatively recently been recognized that the properties of certain finite groups and the structure of certain finite geometries/point-line incidence structures are tied very closely to each other. There exists, in particular, a large family of groups relevant for physics where the (non)commutativity of two distinct elements

can be expressed in the language of finite symplectic polar spaces (the corresponding points (not) being joined by an isotropic line; see, e.g. [1–14]) and/or finite projective ring lines (the corresponding unimodular vectors (not) lying on the same free cyclic submodule; see, e.g. [15–26]). Most recently, invoking also some finite generalized polygons, this link has been employed in [27, 28] to shed a novel light on and get deeper insights into the so-called black hole analogy — a still puzzling formal relation between the entropy of certain stringy black holes and the entanglement properties of some small-level quantum systems (for a recent comprehensive review, see [29]). A concept that played a crucial role in the latter discovery turned out to be that of the Veldkamp space of a point-line incidence structure [30, 31].

The point-line geometry currently central to understanding some particular aspects of the black hole analogy is the unique generalized quadrangle of order $(2, 4)$, $\text{GQ}(2, 4)$ [28]. Its Veldkamp space is isomorphic to $\text{PG}(5, 2)$ [30], which is the natural embedding space not only for $\text{GQ}(2, 4)$ itself, but also for the split Cayley hexagon of order two and the Klein quadric — other two prominent finite geometries linking quantum information with black holes [27]. The geometry $\text{GQ}(2, 4)$ contains several notable subgeometries, one of them being the unique generalized quadrangle of order two, $\text{GQ}(2, 2)$. This quadrangle and its associated Veldkamp space ($\simeq \text{PG}(4, 2)$) were found to underlie the commutation relations between the elements of two-qubit Pauli group and to also clarify some conceptual issues of quantum mechanics [20, 24, 25]. Based on these observations, Vrana and Lévay [31] have even discovered a whole family of Veldkamp spaces, which are associated with the Pauli groups of qubits of arbitrary multiplicity.

To know the structure of a Veldkamp space is vital not only for its own sake, but also because this space fully encodes information about the point-line incidence geometry it is associated with (or generated by). In this paper, with our interest mainly stirred by the above-described remarkable physical applications, we shall focus on the Veldkamp space of the point-line incidence structure that is known under several names: the smallest slim dense near hexagon [32, 33], the 27_3 Gray configuration [34], or, simply, a $(3 \times 3 \times 3)$ -grid (and in what follows denoted as $L_3^{\times 3}$). Although $L_3^{\times 3}$ may seem to be a rather trivial geometry, its Veldkamp space is endowed with a rich and complex structure well worth deserving a closer look at.

Brouwer *et al.* [32] proved that there are 11 isomorphism types of slim dense near hexagons. Of these 11, the near hexagons of sizes 27, 45 and 81 are the most relevant for physical applications. This paper is devoted to a study of the first of the three examples; we plan to describe the other two cases separately. However, it is important to emphasize at the very beginning that this paper is *not* about the well-known universal embedding of $L_3^{\times 3}$ into $\text{PG}(7, 2)$. Although it will be shown that our Veldkamp space is indeed isomorphic to $\text{PG}(7, 2)$, the details of the isomorphism are nontrivial, and it is precisely these details that are of interest for envisaged physical applications.

2. Near Polygons, Quads, Geometric Hyperplanes and Veldkamp Spaces

In this section we gather all the basic notions and well-established theoretical results that will be needed in the sequel.

A *near polygon* (see, e.g. [33] and references therein) is a connected partial linear space $S = (P, L, I)$, $I \subset P \times L$, with the property that given a point x and a line L , there always exists a unique point on L nearest to x . (Here distances are measured in the point graph, or collinearity graph of the geometry.) If the maximal distance between two points of S is equal to d , then the near polygon is called a near $2d$ -gon. A near 0-gon is a point and a near 2-gon is a line; the class of near quadrangles coincides with the class of generalized quadrangles.

A nonempty set X of points in a near polygon $S = (P, L, I)$ is called a subspace if every line meeting X in at least two points is completely contained in X . A subspace X is called geodetically closed if every point on a shortest path between two points of X is contained in X . Given a subspace X , one can define a sub-geometry S_X of S by considering only those points and lines of S which are completely contained in X . If X is geodetically closed, then S_X clearly is a sub-near-polygon of S . If a geodetically closed sub-near-polygon S_X is a nondegenerate generalized quadrangle, then X (and often also S_X) is called a *quad*.

A near polygon is said to have order (s, t) if every line is incident with precisely $s + 1$ points and if every point is on precisely $t + 1$ lines. If $s = t$, then the near polygon is said to have order s . A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance two have at least two common neighbors. A near polygon is called *slim* if every line is incident with precisely three points. It is well-known (see, e.g. [35, 36]) that there are, up to isomorphism, three slim nondegenerate generalized quadrangles. The (3×3) -grid is the unique generalized quadrangle of order $(2, 1)$, $\text{GQ}(2, 1)$. The unique generalized quadrangle of order 2, $\text{GQ}(2, 2)$, is the generalized quadrangle of the points and lines of $\text{PG}(3, 2)$ which are totally isotropic with respect to a given symplectic polarity. The points and lines lying on a given nonsingular elliptic quadric of $\text{PG}(5, 2)$ define the unique generalized quadrangle of order $(2, 4)$, $\text{GQ}(2, 4)$. Any *slim dense* near polygon contains quads, which are necessarily isomorphic to either $\text{GQ}(2, 1)$, $\text{GQ}(2, 2)$ or $\text{GQ}(2, 4)$.

Next, a *geometric hyperplane* of a partial linear space is a proper subspace meeting each line (necessarily in a unique point or the whole line). The set of points at nonmaximal distance from a given point x of a dense near polygon S is a hyperplane of S , usually called the *singular* hyperplane with *deepest* point x . Given a hyperplane H of S , one defines the *order* of any of its points as the number of lines through the point which are fully contained in H ; a point of a hyperplane/sub-configuration is called *deep* if all the lines passing through it are fully contained in the hyperplane/sub-configuration. If H is a hyperplane of a dense near polygon S and if Q is a quad of S , then precisely one of the following possibilities

occurs: (1) $Q \subseteq H$; (2) $Q \cap H = x^\perp \cap Q$ for some point x of Q ; (3) $Q \cap H$ is a sub-quadrangle of Q ; and (4) $Q \cap H$ is an ovoid of Q . If case (1), case (2), case (3), or case (4) occurs, then Q is called, respectively, *deep*, *singular*, *sub-quadrangular*, or *ovoidal* with respect to H . If S is slim and H_1 and H_2 are its two distinct hyperplanes, then the complement of symmetric difference of H_1 and H_2 , $\overline{H_1 \Delta H_2}$, is again a hyperplane; this means that the totality of hyperplanes of a slim near polygon form a vector space over the Galois field with two elements, $\text{GF}(2)$. In what follows, we shall put $\overline{H_1 \Delta H_2} \equiv H_1 \oplus H_2$ and call it the (Veldkamp) sum of the two hyperplanes.

Finally, we shall introduce the notion of the *Veldkamp space* of a point-line incidence geometry $\Gamma(P, L)$, $\mathcal{V}(\Gamma)$ [37]. Here, $\mathcal{V}(\Gamma)$ is the space in which: (i) a point is a geometric hyperplane of Γ and (ii) a line is the collection $H'H''$ of all geometric hyperplanes H of Γ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where H' and H'' are distinct points of $\mathcal{V}(\Gamma)$. Following [20, 30], we also adopt here the definition of Veldkamp space given by Buekenhout and Cohen [37] instead of that of Shult [38], as the latter is much too restrictive by requiring any three distinct hyperplanes H', H'' and H''' of Γ to satisfy the following two conditions: (i) H' is not properly contained in H'' and (ii) $H' \cap H'' \subseteq H'''$ implies $H' \subset H'''$ or $H' \cap H'' = H' \cap H'''$. The two definitions differ in the crucial fact that whereas the Veldkamp space in the sense of Shult is *always* a linear space, that of Buekenhout and Cohen needs not be so; in other words, Shult's Veldkamp lines are always of the form $\{H \in \mathcal{V}(\Gamma) \mid H \supseteq H' \cap H''\}$ for certain geometric hyperplanes H' and H'' .

3. Veldkamp Space of $L_3^{\times 3}$

3.1. Geometric hyperplanes

Our point-line geometry $L_3^{\times 3}$ consists of 27 points and the same number of lines, with three points on a line and, dually, three lines through a point. Its full group of automorphisms G is isomorphic to $S_3 \wr S_3$, of order 1296 [32]. As already mentioned, apart from being the smallest slim dense near hexagon, it is also one of a pair of dual to each other 27_3 configurations whose incidence graph is the *Gray* graph [34] — the smallest cubic graph which is edge-transitive and regular, but not vertex-transitive [39]. Being a slim geometry, its geometric hyperplanes are readily found by employing the above-mentioned property that the complement of symmetric difference of any two of its hyperplanes is again a hyperplane. There is a famous result of Ronan [40] stating that if a point-line incidence geometry is embeddable and is slim, then all of its geometric hyperplanes arise from its universal embedding. As our geometry is indeed universally embeddable into $\text{PG}(7, 2)$, we could readily get the hyperplanes from the structure of this projective space. But as the focus of our paper is on the community of physicists for which the concept of Veldkamp space is virtually unknown, we opt below for a less abstract and more elementary, diagrammatical exposition of the topic.

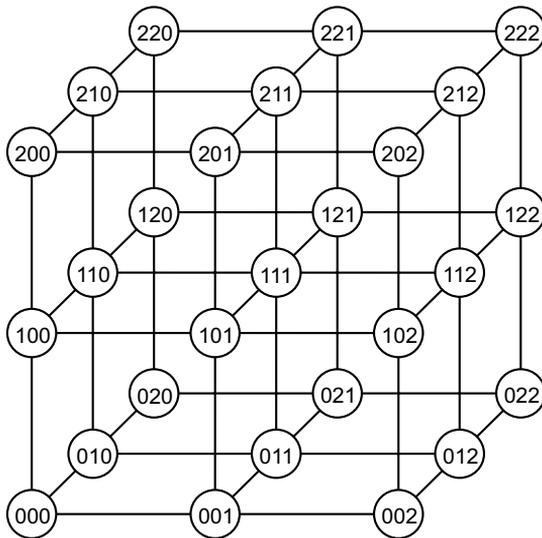


Fig. 1. A diagrammatic illustration of the structure of $L_3^{\times 3}$. The points are represented by circles and the lines by straight segments joining triples of points. Also shown is a ternary digits labeling of the points, employed in the sequel.

To this end, we shall use a diagrammatical representation of our geometry as a $(3 \times 3 \times 3)$ -grid, Fig. 1. From this representation we find that all its quads are $GQ(2, 1)$ s, and there are altogether nine of them. Its singular hyperplane, H_1 , is endowed with 19 points and 15 lines, with 12 points of order two and seven of order three, and three deep and six singular quads — as easily discerned from Fig. 2(a). To find the stabilizer group of this type of geometric hyperplane, one thinks of $L_3^{\times 3}$ as a collection of triples of ternary digits (Fig. 1). We can act on the first, second and third digit independently by permuting by S_3 . Finally, we can permute the digits themselves by another S_3 . The stabilizer of an H_1 is the same as the stabilizer of its deepest point, i.e. $Z_2 \wr S_3$, of order 48; taking this point to be 000, the Z_2 decides whether or not to exchange 1 and 2 in each position and the S_3 permutes the positions. There are altogether $|G|/|Z_2 \wr S_3| = 27$ copies of H_1 in $L_3^{\times 3}$.

Next, pick up two distinct copies $H_1(x)$ and $H_1(y)$ of H_1 , where x and y are the corresponding deepest points. If x and y are at distance two from each other, then $H_1(x) \oplus H_1(y) \equiv H_2$ is a new type of hyperplane, not isomorphic to H_1 . As depicted in Fig. 2(b), an H_2 comprises 15 points and nine lines, with six points of order one and two, and three deep points; it features one deep, six singular and two ovoidal quads. Its stabilizer is $Z_2 \times S_3 \times Z_2$ of order 24. The first Z_2 switches the two planes/quads parallel to the included one (0ab, say). We also need to permute the points 000, 011 and 022 among themselves; this means that we can do one permutation (S_3) affecting the second and third coordinates simultaneously,

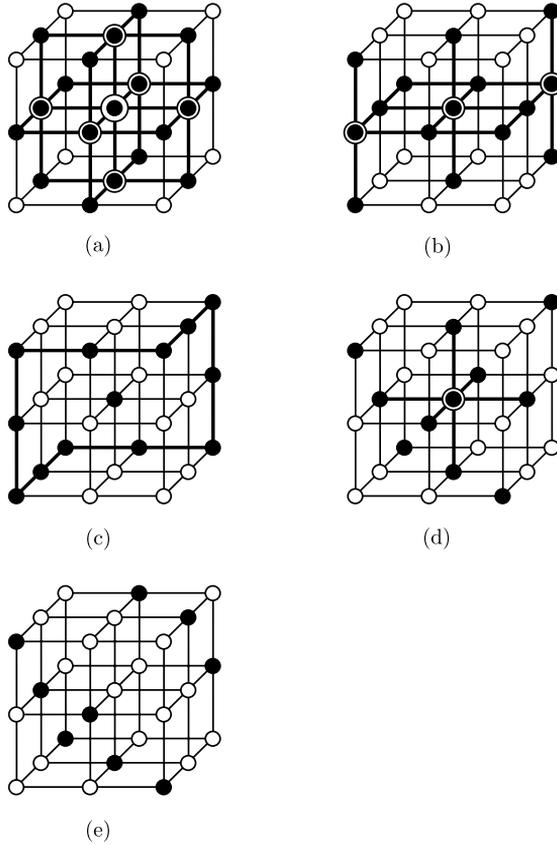


Fig. 2. A diagrammatic illustration of the five distinct types of geometric hyperplanes of the smallest slim dense near hexagon. The points/lines belonging to a hyperplane are boldfaced; double circles stand for deep points.

and also swap the second and third coordinates (that is the second Z_2). We find altogether 54 copies of H_2 in $L_3^{\times 3}$.

If the two deepest points are at maximum distance from each other, then $H_1(x) \oplus H_1(y) \equiv H_3$ is again a new kind of hyperplane, not isomorphic to the previous two — see Fig. 2(c). An H_3 comprises 13 points and six lines, with one point of order zero (henceforth called the nucleus), six points of orders one and two, and no deep points. There are no deep quads there, only six singular and three ovoidal ones. The stabilizer is D_{12} . For if one looks at the copy of H_3 shown in Fig. 2(c) along the appropriate axis, it looks like a regular hexagon with a dot (= the nucleus) in the middle; from this perspective, the result is obvious. Altogether 108 copies of H_3 sit inside $L_3^{\times 3}$.

There exist two more types of geometric hyperplanes. A hyperplane of type H_4 , Fig. 2(d), features 11 points and three lines, four points of order zero, six of order one and one deep point, and, like the previous type, it exhibits only singular (three)

and ovoidal (six) quads. It is stabilized by S_4 . This can be seen from Fig. 2(d), as the four points of order zero form the vertices of a regular tetrahedron and the symmetry group of our H_4 is that of the tetrahedron (including reflections). One can get a copy of H_4 in the following manner. Take a copy of H_2 ; we already know that any H_2 is the complement of the symmetric difference of two H_1 s whose deeps are at distance two from each other. An H_4 is then the complement of the symmetric difference of the H_2 and the copy of H_1 whose deep is at distance three from each of the previous two. Hence, one gets an H_4 as the sum of *three* H_1 s, two of which deeps are at distance two from each other and the third one being at the maximum distance from the two. The cardinality of H_4 is 54.

The final type, H_5 , is an ovoid of $L_3^{\times 3}$, that is a set of nine pairwise noncollinear points (Fig. 2(e)). Hence, in this case all the points are of zeroth order and all the quads are ovoidal. The stabilizer group is $E \times S_3$, where $E \simeq (Z_3 \times Z_3) \times Z_2$, and the Z_2 acts as inversion. There is a nice description of this using the ternary digits method. Take all coordinates whose digits sum to zero mod 3. The S_3 just permutes the coordinates. The group E has order 18 and is generated by an S_3 and an element g of order three. The element g acts by increasing (respectively, decreasing) the second (respectively, third) digit by 1 mod 3. The S_3 acts as the same permutation (value permutation, not place permutation) on each digit simultaneously. To get a copy of H_5 , one selects an H_3 , any of which is the complement of the symmetric difference of two H_1 s whose deeps are at maximum distance from each other. A copy of H_5 is then found as the complement of the symmetric difference of the H_3 and the copy of H_1 whose deep is at maximum distance from either of the two. Hence, one gets an H_5 as the sum of *three* H_1 s whose deeps are pairwise at maximum distance from each other (compare with the preceding case). There are only 12 copies of H_5 inside $L_3^{\times 3}$.

The attentive reader might have noticed that H_1 plays a special role amongst the hyperplanes, because each of the remaining types can be obtained as the sum of several copies of H_1 ; the smallest such number is called the *weight* of a hyperplane. From what we said above, it follows that H_2 and H_3 are of weight two, whereas H_4 and H_5 have weight three. In this respect, $L_3^{\times 3}$ resembles the *dual* of the split Cayley hexagon of order two [41], save for the fact that the latter features many more types of geometric hyperplanes.

All our findings are summarized in Table 1. We see that the total number of hyperplanes is $27 + 54 + 108 + 54 + 12 = 255$. As we already know that hyperplanes form a GF(2)-vector space and $255 = 2^8 - 1 = |\text{PG}(7, 2)|$, one is immediately tempted to conjecture that $\mathcal{V}(L_3^{\times 3})$ is isomorphic to PG(7, 2). In the next section, we shall show that this is indeed so.

3.2. Veldkamp lines

As each Veldkamp line is obviously of the form $\{H', H'', H' \oplus H''\}$, and since $L_3^{\times 3}$ is rather small, diagrams of the form given in Fig. 2 were easy to use to find out “by

Table 1. The types of geometric hyperplanes of the smallest slim dense near hexagon.

Hp	Pts	Lns	# of points of order				# of quads of type			StGr	Wgt	Crd
			0	1	2	3	deep	sng	ovd			
H_1	19	15	0	0	12	7	3	6	0	$Z_2 \wr S_3$	1	27
H_2	15	9	0	6	6	3	1	6	2	$Z_2^{\times 2} \times S_3$	2	54
H_3	13	6	1	6	6	0	0	6	3	D_{12}	2	108
H_4	11	3	4	6	0	1	0	3	6	S_4	3	54
H_5	9	0	9	0	0	0	0	0	9	$E \rtimes S_3$	3	12

hand” all 41 different types of Veldkamp lines, whose basic properties are listed in Table 2. A rather coarse classification in terms of the composition is refined by that of the properties of the cores. To classify each type unambiguously, we have to further refine, in the relevant cases, the core line cardinality entries by stating whether the lines of the core are concurrent (“c”) or parallel/mutually skew (“p”) and specify the core point cardinality entries even in a greater variety of ways, as follows:

- $4_{(3:1)}$ means that one of the points is at maximum distance from the other three, in contrast to $4_{(2:2)}$ where two points are such that either of them is at maximum distance from the remaining three (to distinguish type 35 from type 36);
- $7_{(2/3)}$ or $9_{(2/3)}$ stand for the fact that the two isolated (= zeroth order) points of the core are at distance $2/3$ from each other (to distinguish type 16 from type 17);
- $X_{[n]}$ illustrates the fact that there exist exactly n quads each of which cuts the core in an ovoid, i.e. a triple of pairwise noncollinear points (to distinguish types 25 to 28 from one another); and, finally,
- $5_{(n)}$ tells us that out of five points there are n such that each is coplanar (that is, shares a quad) with any of the remaining four.

Table 2 reveals a number of interesting properties of the Veldkamp lines. First, we notice that there is only one type (41) whose core is an empty set; all the three Veldkamp points of this line are of the same kind, namely H_5 . Next, there are nine different types of Veldkamp lines each of which consists of the hyperplanes of the same type; these are types 1 (H_1), 6, 10, 11 (H_2), 23, 24, 28 (H_3), 37 (H_4) and 41 (H_5). On the other hand, we find nine (out of theoretically possible $10 = \binom{5}{3}$) types featuring three different kinds of hyperplanes; these are types 5, 8, 13, 20, 22, 25, 27, 31 and 39. Interestingly, the only nonexistent “heterogeneous” combination is the H_1 - H_2 - H_5 one. The most “abundant” type of Veldkamp points is H_3 , which occurs in 23 types of lines, followed by H_2 (20), H_4 (17), H_1 (13) and finally by H_5 (9). Remarkably, H_3 also prevails in multiplicity ≥ 2 (12 types), whereas in “singles” the primacy belongs to H_2 (13 types). It is also worth mentioning that there are as many as four different types of the form H_2 - H_3 - H_3 (15, 16, 17 and 19). All these observations are gathered in Table 3.

The structure of Table 3 can easily be demystified if one looks at a decomposition of the group of automorphisms into double cosets with respect to stabilizer

Table 2. The types of Veldkamp lines of the Veldkamp space of the smallest slim dense near hexagon. The details of the symbols/notation are explained in the text.

Type	Core		Composition					Crd
	Pts	Lns	H_1	H_2	H_3	H_4	H_5	
1	15	11	3	—	—	—	—	27
2	13	8	2	1	—	—	—	162
3	12	6	2	—	1	—	—	108
4	11	7	1	2	—	—	—	81
5	10	4	1	1	1	—	—	648
6	9	6	—	3	—	—	—	18
7	9	4	1	—	2	—	—	324
8	9 ₍₂₎	3c	1	1	—	1	—	324
9	9	3	1	—	2	—	—	324
10	9	3p	—	3	—	—	—	18
11	9 ₍₃₎	3c	—	3	—	—	—	108
12	8	3	—	2	1	—	—	648
13	8	2	1	—	1	1	—	648
14	7	3	1	—	—	2	—	27
15	7	2p	—	1	2	—	—	162
16	7 ₍₂₎	2c	—	1	2	—	—	324
17	7 ₍₃₎	2c	—	1	2	—	—	324
18	7 _[2]	1	—	2	—	1	—	162
19	7 _[1]	1	—	1	2	—	—	324
20	7	0	1	—	1	—	1	108
21	7	0	1	—	—	2	—	108
22	6	2c	—	1	1	1	—	648
23	6	2p	—	—	3	—	—	108
24	6	1	—	—	3	—	—	648
25	6 _[3]	0	1	—	—	1	1	216
26	6 _[2]	0	—	2	—	—	1	108
27	6 _[1]	0	—	1	1	1	—	648
28	6 _[0]	0	—	—	3	—	—	36
29	5 _[1]	1	—	1	—	2	—	162
30	5 _[0]	1	—	—	2	1	—	648
31	5 ₍₂₎	0	—	1	1	—	1	324
32	5 ₍₁₎	0	—	1	—	2	—	324
33	5 ₍₀₎	0	—	—	2	1	—	648
34	4	0	—	—	2	—	1	324
35	4 _(3:1)	0	—	—	1	2	—	216
36	4 _(2:2)	0	—	—	1	2	—	324
37	3	1	—	—	—	3	—	54
38	3 _[1]	0	—	1	—	—	2	54
39	3 _[0]	0	—	—	1	1	1	216
40	2	0	—	—	—	2	1	108
41	0	0	—	—	—	—	3	4

Table 3. A succinct overview of the composition of the Veldkamp lines; here, for example, number 6 in the row H_2 and the column H_4 means that there are six different types of lines featuring both H_2 and H_4 .

	H_1	H_2	H_3	H_4	H_5
H_1	3	4	6	5	2
H_2		7	9	6	3
H_3			12	8	4
H_4				8	3
H_5					2

(sub-)groups of hyperplanes. Let us recall (see, e.g. [42]) that given a group G and its two (not necessarily distinct) subgroups K_1 and K_2 , the set

$$K_1 a K_2 \equiv \{k_1 a k_2 \in G \mid a \in G, k_1 \in K_1, k_2 \in K_2\}$$

is called a double coset with respect to K_1 and K_2 . So, for $G \simeq S_3 \wr S_3$ and with K_i , $i = 1, 2$, identified with the stabilizers of five different kinds of hyperplanes, one finds — using, for example, a GAP code [43] — the number of representatives of double cosets for different combinations of hyperplanes as given in Table 4. Comparing the last two tables one sees that they are *almost* identical; the only discrepancies occur in the H_3 - H_3 , H_3 - H_4 and H_4 - H_4 entries, where we always find more double coset representatives than line types. This means that there are particular Veldkamp line types that correspond to more than one double coset. And, indeed, after some work one finds that types 33 and 35 correspond each to two double coset representatives, whereas type 24 corresponds to as many as three. To illustrate the origin of this puzzling “multi-valuedness” property, we shall consider the last-mentioned case.

We first introduce some more terminology. Given a copy of H_3 and disregarding its nucleus, the 12 remaining points are such that there exist exactly two distinct points coplanar with all of them. Let us call this pair of points the “axis” of the

Table 4. The set of representatives, after discarding the relation of equality, of double cosets of the group G of automorphisms of the near hexagon with respect to the stabilizer groups of all five different kinds of hyperplanes. By abuse of notation, the stabilizer group of a hyperplane is denoted by the same symbol as the hyperplane itself. Thus, for example, the entry 6 in the row H_2 and the column H_4 should read that there are six distinct double cosets $H_2 a H_4$, $a \in G$.

	H_1	H_2	H_3	H_4	H_5
H_1	3	4	6	5	2
H_2		7	9	6	3
H_3			15	9	4
H_4				9	3
H_5					2

H_3 . Next, let us select three distinct copies A , B and C of H_3 as follows (employing again our ternary digits representation of the points of the near hexagon): A is the H_3 with nucleus 111 and axis $\{000, 222\}$, B is the H_3 with nucleus 020 and axis $\{102, 211\}$, and C is the H_3 with nucleus 122 and axis $\{001, 210\}$. It is not difficult to verify that the triple $\{A, B, C\}$, with its core comprising the points 012, 020, 111, 200, 201 and 202, is indeed a Veldkamp line of type 24. Now, consider the six *ordered* pairs $\{(A, B), (A, C), (B, A), (B, C), (C, A), (C, B)\}$. The orbits of G acting on this set are $\{(A, B), (B, A)\}$, $\{(A, C), (B, C)\}$, and $\{(C, A), (C, B)\}$ because (A, C) is not G -conjugate to (C, A) even though A is G -conjugate to C ! This just means that the *single* type of a Veldkamp line, that is to say the triple $\{A, B, C\}$, corresponds to *three* distinct double cosets; one of them is symmetric, the other two are transposes of each other.

The remaining issue is to find the cardinality for each type of Veldkamp lines and verify that their sum indeed amounts to 10,795, that is to the number of lines of $\text{PG}(7, 2)$. These calculations were mostly done by computer, again using a particular GAP code [43], and their results are given in the last column of Table 2. It must be stressed that types 6, 23, 37 and 41 behave in a rather peculiar way as in these cases it is impossible to fully reconstruct/recover the hyperplanes of the line from the configuration of the core and, therefore, one has to be more careful in counting here. It is also worth noting that there are only four distinct Veldkamp lines with the empty core, that cardinalities 4, 36 and 81 occur just for a single line type, and that there exist as many as 10 different types of lines of cardinality 324; the next most frequent cardinality is 648, occurring in eight distinct types.

We shall conclude this section by the following important observation. Suppose we have a Veldkamp space over $\text{GF}(2)$ whose associated point-line incidence geometry has an odd number of points and also all its geometric hyperplanes have an odd cardinality. Then, given any two distinct hyperplanes H' and H'' , there exists on this Veldkamp space a G -invariant $\text{GF}(2)$ -bilinear form $B(H', H'')$ defined as follows: $B(H', H'') = 0$ if $|H' \cap H''|$ is odd and $B(H', H'') = 1$ if $|H' \cap H''|$ is even. The G -invariance of the form is immediate, and it turns out that this is the only nontrivial G -invariant bilinear form over $\text{GF}(2)$ on the Veldkamp space. The $\text{GF}(2)$ -bilinearity stems from the facts that: (a) if H' , H'' and H''' form a Veldkamp line then their pairwise intersection is the same as their triple intersection and (b) the total number of points is odd and so is the number of points in each hyperplane. In our case, the corresponding form is the unique symplectic form of $\text{PG}(7, 2)$ with respect to which the totality of 10,795 Veldkamp lines split into two disjoint sets: 5355 isotropic lines (odd core point cardinality) and 5440 nonisotropic ones (even core point cardinality).

4. Summary and Conclusion

A comprehensive description of the Veldkamp space of the smallest slim dense near hexagon (*alias* the Gray configuration, or a $(3 \times 3 \times 3)$ -grid) has been presented.

Being isomorphic to $\text{PG}(7, 2)$, its $2^8 - 1 = 255$ Veldkamp points fall into five distinct classes. Each class is uniquely characterized by the number of points/lines as well as by a sequence of the cardinalities of points of given orders and/or that of (grid-)quads of given types; in addition, we also provide its weight, stabilizer group within the full automorphism group of the near hexagon and the total number of copies. Similarly, the totality of $\binom{255}{2}/3 = 10,795$ Veldkamp lines are shown to form 41 different types. A complete classification of them, partly based on the properties of double cosets of the group of automorphisms with respect to the stabilizer groups of hyperplanes, is given in terms of the properties of their cores and the types of the hyperplanes they are composed of. Given a specific symplectic polarity in $\text{PG}(7, 2)$, we also show that isotropic/nonisotropic Veldkamp lines are of those types whose core point cardinality is odd/even. It is establishing the properties of the Veldkamp lines and their cores, some of which are quite intricate and were rather difficult to discern, which we regard as the main result of the paper. In a similar way to what the Veldkamp lines of $\text{GQ}(2, 2)$ revealed to us about the finer structure of the algebra of a two-qubit Pauli group [20], knowing the structure of the cores of our near hexagon will be vital for getting deeper insights into the commutation algebra of quantum operators/observables underlined by this incidence geometry.

As already stressed, we expect this space to have interesting physical applications. We just give an outline of a couple of arguments giving support to such expectations. It is a well-known fact that the group $\text{Spin}(8)$, the double-cover of $\text{SO}(8)$, has exactly three irreducible real representations of degree eight and, accordingly, three representation spaces: one vectorial and two spinorial. Remarkably, the three spaces are on equal footing and there exists an extra automorphism, known as *triality*, that permutes them. This feature has already been found to play a distinguished role in certain supersymmetry and supergravity theories (see, e.g. [44]). Strikingly, it is also our Veldkamp space that is endowed with a *triality*, albeit a geometric one. To be more specific (see, e.g. [45]), let us consider a nonsingular hyperbolic quadric $Q^+(7, 2)$ in $\text{PG}(7, 2)$. The generators of this particular quadric are three-dimensional subspaces. The set of generators can be divided into two families P_1 and P_2 such that two distinct generators belong to the same family if and only if they intersect in a subspace of odd dimension (i.e. in the empty set, a line or a three-dimensional subspace (in which case they coincide)). Let P_0 be the set of points of $Q^+(7, 2)$ and call the elements of P_i the i -points, $i \in \{0, 1, 2\}$; let L denote the set of lines on $Q^+(7, 2)$. A simple counting argument yields $|P_0| = |P_1| = |P_2|$. A *triality* of $Q^+(7, 2)$ is a bijection μ that maps: $P_0 \mapsto P_1$, $P_1 \mapsto P_2$, $P_2 \mapsto P_0$, $L \mapsto L$, is incidence preserving and satisfies $\mu^3 = 1$. An absolute i -point is an i -point that is incident with its image under the triality μ , $i \in \{0, 1, 2\}$; an absolute line is a line that is fixed by μ . For certain trialities the incidence structure formed by the absolute i -points and the absolute lines, with given incidence, is a generalized hexagon. And for a *specific* triality we get the *split Cayley hexagon of order two* — an important representative of finite geometry found to underlie the properties of entropy formulas of a distinguished class of stringy black holes [27].

The second argument is related to the concept of superqubits as proposed in a very recent paper by Borsten *et al.* [46]. Here, the structure of $L_3^{\times 3}$ seems to be *directly* related to the Hilbert space of three-superqubits. This super-Hilbert space is 27-dimensional and associating bijectively its dimensions with the 27 points of $L_3^{\times 3}$ one immediately finds that its 13 fermionic dimensions should correspond to a geometric hyperplane of type H_3 and its 14 bosonic ones to the complement of the hyperplane in question. Moreover, any truncation to two-superqubits would correspond to a GQ(2,1)-quad in $L_3^{\times 3}$, and the associated splitting of the nine-dimensional super-Hilbert sub-space into five boson and four fermion dimensions would simply correspond to a perp-set (of the GQ(2,1)) and its complement, respectively. A firmer footing for this argument can only be given after we are familiar with the structure of the Veldkamp spaces of other two closely-related point-line incidence geometries, namely of $L_3 \times \text{GQ}(2, 2)$ and $L_3 \times \text{GQ}(2, 4)$, in each of which our near hexagon sits as a subgeometry. We are almost done with the first case [47], but still far from being so for the second geometry.

Further explorations along these lines are clearly vital as they may, among other things, shed further light on the relation between quantum information theory and physics of stringy black holes outlined in the introduction. In view of these facts, proper understanding of the structure of $\mathcal{V}(L_3^{\times 3})$ — as expounded in preceding sections — is of great importance.

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