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Temporal Dimension over Galois Fields of Characteristic Two

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Abstract—This paper highlights the most remarkable properties of pencil-generated temporal dimensions over Galois fields of even characteristic (GF(2^n)). It is shown that in the field of real numbers the ordinary arrow of time emerges in GF(2^n) either as a peculiar temporal arrow featuring the domains of the past and future, but lacking the moment of the present, or as an observable *spatial* dimension. In order to understand these peculiarities more fully a quite detailed account is presented of the properties of a point-conic in projective planes over Galois fields of both even and odd order.

1. INTRODUCTION

This paper is the third in a series devoted to the exposition of the fundamentals of an algebro-geometric theory of the arrow of time based on pencils of conics in a projective plane [1–4].

In the first paper of the series [3], we gave a sufficiently exhaustive account of the theory in a projective plane over reals. The exposition started with a succinct outline of the mathematical formalism and went on with introducing the definitions of pencil-time and pencil-space, both at the abstract (projective) and concrete (affine) levels. The structural properties of all possible types of temporal arrows were analysed and, based on symmetry principles, the uniqueness of that which best mimicked the reality was justified. A profound connection between the character of different 'ordinary' arrows and the number of spatial dimensions was also revealed.

In the second paper [4], we made an important step by extending the concept of pencil-generated temporal dimensions to the planes over arbitrary ground fields not of characteristic two. It was found that if the ground field is algebraically closed, every temporal dimension is devoid of any notion of the future. We further learned that the question of the existence of (any of the) three temporal domains—the past, present and future—is completely meaningless unless we *a priori* specify the character of the ground field; what, for example, is the *future* event over reals might well be found to belong to the domain of the *past* after switching to some other field(s), such as F_7—the (finite) field of residues modulo 7. It was argued that these findings may provide an interesting insight into a number of puzzling phenomena related to 'psychological' time like clairvoyance, precognition, déjà-vu experiences, etc., once we adopt the hypothesis that our (sub)conscious is capable of operating at various levels corresponding to different ground fields.

In order to make our analysis complete, it still remains for us to examine the structure of the temporal in projective planes whose ground fields are of characteristic two. And this is the subject of the present paper.

2. BASIC PROPERTIES OF GALOIS FIELDS

We begin by briefly reviewing some of the necessary background concerning the finite generalizations of the concept of number, known to mathematicians as *Galois fields*; the reader who is interested in further details should consult, for example, [5–9].

A set of elements F that is closed under two operations called addition and multiplication is a *field* if:

1. it is an Abelian (i.e., consisting of commuting elements) group under addition with identity 0;
2. all the elements, excluding zero, form an Abelian group under multiplication with identity 1; and
3. $u(v+w) = uv+uw$, $(u+v)w = uw+vw$ for all u, v and $w \in \mathsf{F}$.

Since the elements of a field F form a group under addition, every element $w \in \mathsf{F}$ has an additive order, namely, the least positive integer r for which $rw = 0$. This additive order, being the same for all non-zero elements of F, is called the *characteristic* of F; it is said to be zero if all non-zero elements have infinite order. If F has a finite characteristic, this must be a prime p. The number of elements of a field with a finite characteristic p is a power p^n, n being a positive integer. For every prime p and every positive integer n, there exists a field having exactly p^n elements; this field is called the *Galois field* of order p^n and is denoted by GF(p^n). Every element w of GF(p^n) satisfies the equation

$$w^{p^n} - w = 0, \tag{1}$$

i.e., the set of elements of the field is formed by the zeros of the polynomial $w^{p^n} - w$. For any field GF(p^n), there exists an element s, called the primitive root, such that

$$\mathrm{GF}(p^n) = \{0,1,s,s^2,\ldots,s^{p^n-2}|s^{p^n-1} = 1\}. \tag{2}$$

At this point, we have at hand a sufficient piece of information in order to reveal the fundamental distinction between the properties of Galois fields of even ($p = 2$) and odd ($p > 2$) characteristic.

Thus, for $p = 2$, eqn (1) reads

$$w^{2^n} - w = 0 \tag{3}$$

and it follows from there that every element of GF(2^n) is a square,

$$w = (w^{2^{n-1}})^2 \tag{4}$$

Moreover,

$$0 = 2w = w + w \Leftrightarrow w = -w, \tag{5}$$

which implies that an element of GF(2^n) has only one square root.

The above described situation is diametrically different from the case $p > 2$ for, among the $p^n - 1$ non-zero elements of a field of odd order, there are $(p^n - 1)/2$ non-squares and the same number of squares—the latter being the even powers of the primitive element s. To see this, we first notice that the primitive element itself satisfies eqn (1), so that

$$s^{p^n-1} = 1, \tag{6}$$

and, as a consequence,

$$s^{(p^n-1)/2} = -1. \tag{7}$$

Now, for the element s^m (m being a positive integer) to be a square, i.e., $s^m = w^2$, we find, from eqn (1), the constraint

$$+1 = w^{p^n-1} = (w^2)^{(p^n-1)/2} = (s^m)^{(p^n-1)/2} = (s^{(p^n-1)/2})^m = (-1)^m, \tag{8}$$

which is indeed met only for m even.

We will conclude this section by having a brief look at the properties of the binomial theorem over finite fields. As is well-known, the binomial theorem describes how to expand the sum of two elements u and v raised to a positive integer power $m \geq 1$

$$(u+v)^m = u^m + \sum_{k=1}^{m-1} \frac{m!}{k!(m-k)!} u^{m-k} v^k + v^m. \tag{9}$$

Here, it is crucial to observe that, if m is a prime, then it divides $m!/k!(m-k)!$ for all k, $1 \leq k \leq m-1$, so that $(u+v)^m = u^m + v^m \pmod p$; hence, in a field of characteristic p, the binomial theorem simply reads

$$(u+v)^p = u^p + v^p. \tag{10}$$

As an interesting corollary, we have

$$(u+v+\ldots+w)^p = u^p + v^p + \ldots + w^p. \tag{11}$$

Having compiled some basic facts about finite fields, we are now able to examine the structure of an arrow of time over GF(2^n).

3. TEMPORAL OVER GALOIS FIELDS OF CHARACTERISTIC TWO

In this section, our primary attention will be focused on the quadratic equation

$$ax^2 + bx + \vartheta = 0, a \neq 0, \tag{12}$$

whose roots also carry, among other things, information about intersection properties of the conics of the pencil†

$$Q^{\vartheta}_{\check{x}\check{x}}(q) \equiv \sum_{i,j=1}^{3} q_{ij}(\vartheta_{1,2})\check{x}_i\check{x}_j = \vartheta_1 \check{x}_1 \check{x}_2 + \vartheta_2 \check{x}_3^2 = 0, \tag{13}$$

with the line

$$\check{x}_1 - a\check{x}_2 - b\check{x}_3 = 0 \tag{14}$$

if $\vartheta \equiv \vartheta_2/\vartheta_1$ and $x \equiv \check{x}_2/\check{x}_3$ [1, 3, 4].

In the case of a field F of odd and/or zero characteristic, the task of finding the zeros of eqn (12) is quite easy as this equation can be completed to the square

$$\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{(2a)^2} = 0, \;\; \Delta \equiv b^2 - 4a\vartheta \tag{15}$$

† The projective symbols and notation are the same as in our previous papers [1–4].

from which it follows that (Theorem 1) it has:

— one (double) root if $\Delta = 0$;
— two distinct roots if Δ is a square in F; and
— no solutions if Δ is a non-square in F.

It is, however, immediately evident that this strategy cannot be employed for fields of characteristic two, due to the presence of modulus 2 that makes eqn (15) singular. The strategy pursued in this case is as follows (see, for example, [7], pp. 3–4]). If $b = 0$, then eqn (12) has just a single solution,

$$x = \sqrt{\frac{\vartheta}{a}}, \tag{16}$$

due to eqn (5). For $b \neq 0$, we introduce the new variables,

$$z \equiv \frac{a}{b}x, \Theta \equiv \frac{a}{b^2}\vartheta, \tag{17}$$

in which eqn (12) acquires the form

$$z^2 + z + \Theta = 0. \tag{18}$$

Next, we define

$$D(w) = w + w^2 + w^4 + \ldots + w^{2^{n-1}}, \tag{19}$$

and, exploiting eqns (3) (5) and (11), verify that

$$D^2(w) + D(w) = 0, \tag{20}$$

that is

$$D(w) = 0, \text{or } D(w) = 1. \tag{21}$$

On the other hand, we notice that

$$\begin{aligned} D(z^2+z+\Theta) &= (z^2+z+\Theta)+(z^2+z+\Theta)^2+\ldots+(z^2+z+\Theta)^{2^{n-1}} \\ &= (z^2+z+\Theta)+(z^4+z^2+\Theta^2)+\ldots+(z^{2^n}+z^{2^{n-1}}+\Theta^{2^{n-1}}) \\ &= (z^2+\ldots+(z^2)^{2^{n-1}})+(z+\ldots+z^{2^{n-1}})+(\Theta+\ldots+\Theta^{2^{n-1}}) \\ &= D(z^2)+D(z)+D(\Theta) \end{aligned} \tag{22}$$

and

$$D^2(z) \equiv (z+z^2+\ldots+z^{2^{n-1}})^2 = (z^2)+(z^2)^2+\ldots+(z^2)^{2^{n-1}} = D(z^2), \tag{23}$$

so that

$$D(z^2+z+\Theta) = D^2(z)+D(z)+D(\Theta) = D(\Theta). \tag{24}$$

Since $D(0) = 0$, the last expression implies that eqn (18) can only be solved if

$$D(\Theta) = 0. \tag{25}$$

Furthermore, it is a simple matter to check that, if $z = z_0$ is a solution to eqn (18), then so is $z_0 + 1$; so really

$$(z_0+1)^2+(z_0+1)+\Theta = z_0^2+1+z_0+1+\Theta = z_0^2+z_0+\Theta = 0, \tag{26}$$

where we have again used eqn (5). Summarizing, we see that (Theorem 2) eqn (18) has, in GF(2^n):

— two distinct roots if $D(\Theta) = 0$, and
— no solutions if $D(\Theta) = 1$.

Let us now return to geometry, eqns (13) and (14), and take the line from eqn (14) as the 'line at infinity'. Theorem 1 then gives us information concerning the affine structure of the pencil (eqn (13)) in the projective planes whose ground fields are of odd and/or zero characteristic, whereas Theorem 2 provides us with the same sort of knowledge for the planes over fields of even characteristic. After recalling our fundamental postulate that the conic corresponding to the parameter ϑ which induces two distinct, one double or no roots of eqn (12) stands, respectively, for the events of the past, present and future [1–4], the above two theorems also inform us about the structure of the arrows of time generated by the corresponding affine images of pencil eqn (13). And here we reveal a very remarkable fact that *what is in the real* (*as well as in any odd*) *plane*, *an ordinary arrow of time*, i.e., the arrow that features the region of the past (Δ-squares), the domain of the future (Δ-non-squares), as well as a single moment of the present ($\Delta = 0$) and which is represented by those affine images of pencil eqn (13) where [1, 3]

$$a \neq 0 \neq b, \tag{27}$$

emerges in even planes as a 'defective' temporal arrow comprising only (*the domains of*) *the past* ($D(\Theta) = 0$) *and future* ($D(\Theta) = 1$). This is a very peculiar kind of temporal defect not encountered in the real plane and so, as to fully comprehend its origin, it is necessary to examine the properties of a conic in even projective planes.

4. A CONIC IN A PROJECTIVE PLANE OVER FIELDS OF EVEN ORDER

To this end we will consider a quadratic from

$$Q_{\check{x}\check{x}}(c) \equiv +c_{11}\check{x}_1^2+c_{22}\check{x}_2^2+c_{33}\check{x}_3^2+c_{12}\check{x}_1\check{x}_2+c_{13}\check{x}_1\check{x}_3+c_{23}\check{x}_2\check{x}_3 = 0. \tag{28}$$

This equation represents a conic, Q, in a projective plane over any ground field (see, for example, [7]). A conic can be either proper or degenerate, depending on whether it, respectively, does not or does contain some singular point(s), i.e., the point(s) at which it has no tangent(s).

Let us find a criterion for the singularity of a conic. It is worth noting that the criterion we have used in the real and odd projective planes, namely $\det a_{ij} = 0$ where

$$\begin{aligned} a_{ij} &= c_{ij}, & i &= j, \\ a_{ij} &= a_{ji} = \frac{1}{2}c_{ij}, & i &\neq j, \end{aligned} \tag{29}$$

apparently cannot be extended to the case of GF(2^n). Hence, in order to find the criterion that would be equally applicable to fields of both characteristics, we have to follow a different route [7].

We shall consider the coordinates ζ_i of a line that is tangent to the conic Q at $\check{x}_i = \check{z}_i$,

$$\varrho\zeta_i = \left.\frac{\partial Q_{\check{x},\check{x}}}{\partial \check{x}_i}\right|_{\check{x}_i=\check{z}_i} \equiv \frac{\partial Q_{\check{z},\check{z}}}{\partial \check{z}_i}, \varrho \neq 0, \tag{30}$$

or, explicitly

$$\varrho\zeta_1 = \frac{\partial Q_{\check{z},\check{z}}}{\partial \check{z}_1} = 2c_{11}\check{z}_1 + c_{12}\check{z}_2 + c_{13}\check{z}_3,$$
$$\varrho\zeta_2 = \frac{\partial Q_{\check{z},\check{z}}}{\partial \check{z}_2} = c_{12}\check{z}_1 + 2c_{22}\check{z}_2 + c_{23}\check{z}_3, \tag{31}$$
$$\varrho\zeta_3 = \frac{\partial Q_{\check{z},\check{z}}}{\partial \check{z}_3} = c_{13}\check{z}_1 + c_{23}\check{z}_2 + 2c_{33}\check{z}_3.$$

If $\check{z}_i$ is a singular point, then

$$\zeta_1 = \zeta_2 = \zeta_3 = 0, \tag{32}$$

and this implies that

$$0 = \det\begin{pmatrix} 2c_{11} & c_{12} & c_{13} \\ c_{12} & 2c_{22} & c_{23} \\ c_{13} & c_{23} & 2c_{33} \end{pmatrix} \equiv 2\beta, \tag{33}$$

where

$$\beta \equiv 4c_{11}c_{22}c_{33} + c_{12}c_{13}c_{23} - c_{12}^2c_{33} - c_{13}^2c_{22} - c_{23}^2c_{11}. \tag{34}$$

Hence, for the planes over GF(p^n), $p > 2$, the condition that the conic given by eqn (28) is singular reads

$$\beta = 0. \tag{35}$$

As for the $p = 2$ case, here we observe that from Euler's theorem,

$$2Q_{\check{z},\check{z}} = \sum_{i=1}^{3} \check{z}_i \frac{\partial Q_{\check{z},\check{z}}}{\partial \check{z}_i}, \tag{36}$$

it follows that the point $\check{z}_i$, for which all $\partial Q_{\check{z},\check{z}}/\partial \check{z}_i$ vanish, does not necessarily lie on Q. In this case, however, eqn (32) is equivalent to

$$0 = 0 + c_{12}\check{z}_2 + c_{13}\check{z}_3,$$
$$0 = c_{12}\check{z}_1 + 0 + c_{23}\check{z}_3, \tag{37}$$
$$0 = c_{13}\check{z}_1 + c_{23}\check{z}_2 + 0,$$

and this set is satisfied by the point $\check{z}_i^0$,

$$\varrho\check{z}_i^0 = (c_{23}, c_{13}, c_{12}),\ \varrho \neq 0, \tag{38}$$

which belongs to Q only if

$$0 = Q_{\check{z}^0,\check{z}^0} = c_{11}c_{23}^2 + c_{22}c_{13}^2 + c_{33}c_{12}^2 + c_{12}c_{23}c_{13} + c_{13}c_{23}c_{12} + c_{23}c_{13}c_{12}$$
$$= c_{11}c_{23}^2 + c_{22}c_{13}^2 + c_{33}c_{12}^2 + c_{12}c_{23}c_{13}, \tag{39}$$

where, again, we have made use of eqn (5). However, the second line of the last equation can, for even fields, be easily recognized to coincide with β and we can thus conclude that eqn (35) represents the singularity condition for conics over arbitrary ground fields.

Let us consider a $p = 2$ conic Q that is proper, $\beta \neq 0$. From eqn (39), it is obvious that $\check{z}_i^0$ cannot lie on such a conic. Further, eqns (31), (38) and (5) imply that the expression

$$\sum_{i=1}^{3} \zeta_i(\check{z}_k)\check{z}_i^0 = \zeta_1(\check{z}_k)\check{z}_1^0 + \zeta_2(\check{z}_k)\check{z}_2^0 + \zeta_3(\check{z}_k)\check{z}_3^0$$
$$= (c_{12}\check{z}_2 + c_{13}\check{z}_3)c_{23} + (c_{12}\check{z}_1 + c_{23}\check{z}_3)c_{13} + (c_{13}\check{z}_1 + c_{23}\check{z}_2)c_{12} \quad (40)$$
$$= 2c_{12}c_{13}\check{z}_1 + 2c_{12}c_{23}\check{z}_2 + 2c_{13}c_{23}\check{z}_3$$

vanishes over GF(2^n), irrespective of the values attained by $\check{z}_i$, hence, also for every point $\check{z}_i \in$Q. This, however, means that the point $\check{z}_i^0$ is shared by all tangents to Q. In other words, we find that *in a projective plane over fields of characteristic two the tangents to a proper point-conic form a pencil*; with the conic defined via eqn (28) the carrier of the pencil, also called the *nucleus*, has the coordinates given by eqn (38) ([7], pp. 144–15). And it is precisely this property, having no analogue in the real and/or odd projective planes, that is 'responsible' for the absence of the present in even temporal arrows.

5. WHY ARE EVEN TEMPORAL ARROWS LACKING THE PRESENT?

An explicit way of seeing this consists of noticing that eqn (13) is a special case of eqn (28), with only the non-zero coefficients being

$$c_{12} = \vartheta_1, c_{33} = \vartheta_2, \quad (41)$$

in order for us to find that ($\varrho \neq 0$)

$$\varrho\check{z}_i^0 = (0,0,\vartheta_1) \quad (42)$$

is the meet of *all* tangents to *any* proper conic of pencil eqn (13). Hence, a given proper conic of the pencil represents the event of the present, i.e., has the ideal line as its tangent, only if $\check{z}_i^0$ satisfies eqn (14),

$$b\vartheta_1 = 0. \quad (43)$$

Since ϑ_1 is, as implied by eqns (35) and (41), a non-vanishing quantity for proper conics, the above equation is equivalent to

$$b = 0, \quad (44)$$

this, however, is in contradiction to the second of of the inequalities (27), which are the necessary conditions for the (very) existence of the arrows of time over the ground field of reals ([3], Sections 3 and 4).

However, even if we sacrificed real temporal arrows and thereby allowed the ideal line to pass via the point $\check{z}_i^0$, (eqn (42)), we would not get any GF(2^n) arrow. The reason for this is that, for $b = 0$, eqn (12) has, as we have already shown, just a single root, that given by eqn (16), for *all* non-singular conics; hence, all proper affine conics are parabolas and the corresponding affine image of pencil eqn (13) is thus homogeneous.

6. R-TEMPORAL AS GF(2^n)-SPATIAL

The above-described 'peculiarities' of the properties of (the pencil of) conics and, consequentially, temporal dimensions in even projective planes can further be sharpened by observing from eqns (35), (34) and (5) that, for a conic to be non-singular over GF(2^n), it is necessary that at least one of the 'skew' c_{ij}, $i \neq j$, deviates from zero. This observation, when looked at from a more general point of view, is found to have a serious physical implication for it sheds very intriguing and completely unexpected light on the relation between time and space.

To make this assertion more explicit, we will again consider our 'favourable' eqn (13). As is immediately obvious from eqns (41), (34) and (35), this equation defines one and the same type of pencil of conics, namely that endowed with two degenerates,

$$\vartheta_1 = 0 \text{ and } \vartheta_2 = 0, \tag{45}$$

irrespective of the character of the ground field, thus also over GF(2^n)—although in the latter case, as we saw in Section 3, the structure of the corresponding temporal dimension is rather odd. Now, let us consider a transformation $\breve{x}_i \rightarrow \breve{y}_i$

$$\varrho \breve{x}_i = \sum_{j=1}^{3} A_{ij} \breve{y}_j, \varrho \neq 0, \tag{46}$$

where

$$A_{ij} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{47}$$

as

$$A \equiv \det A_{ij} = -2, \tag{48}$$

this transformation is non-singular in fields of $p > 2$ and/or zero characteristic and can thus be regarded as *an allowable coordinate* transformation in the corresponding projective planes (see, for example, [10], Chap. V). Hence, in such planes, $\breve{y}_i$ represent the new system of coordinates, in which eqn (13) acquires a purely diagonal form,

$$\vartheta_1 \breve{y}_1^2 - \vartheta_1 \breve{y}_2^2 + \vartheta_2 \breve{y}_3^2 = 0, \tag{49}$$

i.e.,

$$c_{11} = -c_{22} = \vartheta_1, c_{33} = \vartheta_2, c_{ij} = 0 \text{ for } i \neq j. \tag{50}$$

Otherwise stated, eqns (13) and (49) are nothing but two perfectly equivalent representations of one and the same pencil of conics, as long as the projective plane is backed by fields not of characteristic two.

This, however, differs from the very root(s) from the situation characterizing the $p = 2$ case. Here transformation eqns (46) and (47) becomes singular ($A = -2 = 0$), and we infer from eqns (34), (50) and (5) that $\beta = 0$ for any ϑ_1 and ϑ_2. In other words, in this case, *all* the conics of the pencil defined by eqn (49) are degenerate, each comprising a (double) real line as we can easily see by rewriting eqn (49) with the help of eqn (11) as

$$(\sqrt{\vartheta_1} \breve{y}_1 - \sqrt{\vartheta_1} \breve{y}_2 + \sqrt{\vartheta_2} \breve{y}_3)^2 = 0, \tag{51}$$

which is equivalent to

$$\sqrt{\vartheta_1} \breve{y}_1 - \sqrt{\vartheta_1} \breve{y}_2 + \sqrt{\vartheta_2} \breve{y}_3 = 0, \tag{52}$$

this indeed represents (see, for example, [3], Section 2) a pencil of *lines* $\zeta_i(\vartheta_{1,2})$,

$$\varrho \zeta_i(\vartheta_{1,2}) = (\sqrt{\vartheta_1}, -\sqrt{\vartheta_1}, \sqrt{\vartheta_2}), \varrho \neq 0, \tag{53}$$

with the meet $\check{y}_i^0$,

$$\varrho\check{y}_i^0 = (1,1,0), \varrho \neq 0. \quad (54)$$

Since, in our theory, pencils of lines are taken to define *spatial* dimensions ([1–3]), this means that starting with the 'diagonal' representation (eqn (49)) of what in fields of odd and/or zero characteristic is a genuine *temporal* dimension (cf. Ref. [3], Section 3 for a detailed description of the $\mathsf{F} = \mathsf{R}$ case), we arrive at a *spatial* coordinate as soon as we switch to fields of characteristic two! Moreover, it presents no difficulty to verify that this spatial dimension is an *observable* one [1–3], because pencil eqn (53) incorporates the ideal line; thus, applying the transformation given by eqns (46) and (47) on eqn (14) yields.

$$(1-a)\check{y}_1-(1+a)\check{y}_2-b\check{y}_3 = 0, \quad (55)$$

whose left-hand side, for $\check{y}_i^0$, looks like

$$(1-a)-(1+a) = -2a \quad (56)$$

and this is indeed (cf. eqn (5)) found to vanish regardless of the values of a and b.

7. GF(2^s) 'SPATIALIZATION' OF TIME—ANOTHER REMARKABLE SIMILARITY WITH CANTORIAN SPACE-TIME

Rephrased in 'plain' English, these latest findings simply tell us that, contrary to the situation in projective planes over ground fields of odd and/or zero characteristic, there is no clear-cut distinction between time and space at the GF(2^s)-level of physical reality. This, perhaps at first sight slightly contraintuitive feature, is, however, another point at which our theory bears strong resemblance to Cantorian space-time [11–16], where time also becomes indistinguishable from the spatial coordinate, there due to its statistical behaviour [14–16]. Although at this stage of the development of the theory, it is hard to see what lies behind such an amazing similarity between the two concepts, we nevertheless have a strong feeling that it is 'more than just a mathematical artefact and that a deeper reality may lurk behind the mathematical veil' [15].

8. CONCLUSION—WHAT ARE THE LIMITS OF THE MIND?

We conclude this paper by some speculative remarks and return to our concept of the mind as an 'entity' capable of operating over an arbitrary field [4]. From what we have found in the preceding sections, it is obvious that the states of (sub)consciousness corresponding to fields of characteristic two represent the most radical departure from our waking state, which is assumed to be based on the field of real numbers. Thus, it is only in the $p = 2$ states when one can experience such bizarre things as the presentless temporal (Section 3), or even a sort of metamorphosis of time into a spatial coordinate (Section 6). This, when also taking into account the findings of [1–4], is the point where our account can deliberately be slanted towards what is certainly a revolutionary idea—that the mind of each of us is not confined to the limits of conventional time and space and what we perceive in our 'normal' state is only a tiny fraction of the world which we all have potential access to. There have, of course, been many attempts to try to understand the brain mechanism in connection with visual hallucinations, for instance at the Santa Fe Institute [18]. However, let us now speak of those who may have had a glimpse at a part of this strange world ([17], pp. 9 and 187):

> The first three hours of my LSD session were experienced as a fantastic battle between the forces of Light and Darkness.... It was fought on all conceivable levels—in the cells and tissues of my body, on the surface of our planet

throughout history, in the cosmic space, and on a metaphysical, transcendental level...After this battle was over, I found myself in a rather unusual state of mind; I felt a mixture of serenity and bliss with the naive and primitive faith of the early Christians. It was a world where miracles were possible, acceptable and understandable. I was preoccupied with the problems of time and space and the insoluble paradoxes of infinity and eternity that baffle our reason in the usual state of consciousness. I could not understand how I could have let myself be 'brainwashed' into accepting the simple-minded concept of one-dimensional time and three-dimensional space as being mandatory and existing in objective reality. It appeared to me rather obvious that there are no limits in the realm of spirit and that time and space are the arbitrary constructs of the mind. Any number of spaces with different order of infinities could be deliberately created and experienced. A single second and eternity seemed to be freely interchangeable. I thought about higher mathematics and saw deep parallels between various mathematical concepts and altered states of consciousness...

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