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Pencils of Conics: a Means Towards a Deeper Understanding of the Arrow of Time?

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Abstract—The paper aims at giving a sufficiently complex description of the theory of pencil-generated temporal dimensions in a projective plane over reals. The exposition starts with a succinct outline of the mathematical formalism and goes on with introducing the definitions of pencil-time and pencil-space, both at the abstract (projective) and concrete (affine) levels. The structural properties of all possible types of temporal arrows are analyzed and, based on symmetry principles, the uniqueness of that mimicking best the reality is justified. A profound connection between the character of different 'ordinary' arrows and the number of spatial dimensions is revealed. © 1998 Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

The depth of our penetration into the problem is measured not by the state of things on the frontier, but rather how far away the frontier lies. One of the most pronounced examples of such a situation concerns, beyond any doubt, the nature of time. What we have precisely in mind here is a "puzzling discrepancy between our perception of time and what modern physical theory tells us to believe" [1], see also [2–6].

Every human being (except for, perhaps, a very young child) is aware of the 'passage' of time. This fact is embodied in the existence of such notions as 'the past', 'the future', the two domains being separated from each other by the single moment of 'the present'. While the past is regarded as fixed, definite, the future is viewed as unknown, uncertain, undetermined; the only perceivable is the present, the 'now'—the ever-changing point 'moving' from the past into the future. Physics, on the contrary, tells us a completely different story. Not only are the vast majority of physical laws time reversible, but the concept of the now itself has no place at all in the temporal of physics. In other words, the equations of physics do not distinguish between the past and future, and seem to be completely ignorant to the very idea of the present.

However, no account of the physical world in its totality can be final which leaves such severe discrepancies between the two concepts of the temporal quite disregarded. Hence, as an attempt to shed some light on this gap we recently came up with the idea of pencil-generated temporo-spatial dimensions [7, 8]. In the first short paper [7] we outlined the fundamentals of the concept, central to which is a projective plane with a prescribed pencil of conics in it. The basis for the abstract 'time' coordinate is represented by the non-degenerate conics of the pencil, while any pencil of lines through a point on a degenerate conic (an s-pencil) serves as a potential 'spatial' dimension. The observed (physical) space-time then appears as a result of the affinization of the projective plane. On the one hand, this produces a spontaneous 'stratification' of the conics in the pencil, thus endowing time with its 'arrow'. On the other hand, it singles out from the set of all the s-pencils those incorporating the ideal line, which are taken to represent the 'observed' spatial dimensions.

On the example of the pencil of conics having three degenerates and no real base points [8], we have further demonstrated that the 'perfect' arrow of time is only found in those cases where the ideal line is free of singular points.[1] It was shown in detail that by shifting the ideal line from its original position the structure of the arrow can continuously be deformed, even to the extent that it completely ceases to exist. This happens at the moment when the ideal line gets in touch with at least one of the singular points, and is accompanied by the fact that either the domains of the past and present, or those of the future and present, collapse to a point.

The aim of the present paper is as follows. We start with a review of the necessary mathematical background and continue by an introduction of the pencil-definitions of both time and space. To fix the ideas, the properties of a space-time configuration generated by the pencil of conics endowed with two singular conics and two base points are described, unearthing a very deep connection between the three-dimensionality of space and the (very) existence of the arrow of time. As the next step, the remaining types of pencils of conics exhibited by a real projective plane are analyzed, and the existence of other two types of one-dimensional temporal arrows is revealed. Specific symmetry constraints are then invoked to decide which of the arrows describes best the observed one. Finally, in the light of these constraints the question of the dimensionality of space is revisited.

2. A BRIEF OUTLINE OF THE FORMALISM

The keystone of our theory of time and space is the notion of a projective space of dimension two, i.e. of a projective plane, together with specified aggregates of geometrical objects lying in it.

A projective plane (over the field of real numbers) is a geometric entity whose points are represented by all real triples $(\check{x}_1, \check{x}_2, \check{x}_3)$, excluding $(0, 0, 0)$, with proportional triples representing the same point. There is a great variety of different types of geometric structures contained in a projective plane, but for our further purposes we will restrict ourselves to the simplest, i.e. linear and quadratic ones. A linear geometric element is the one defined by the equation

$$L_{\zeta\check{x}} \equiv \sum_{i=1}^{3} \zeta_i \check{x}_i = 0, \tag{1}$$

and represents either a (straight) 'line', with ζ_i being constant and $\check{x}_i$ variable, or 'a pencil of lines' for ζ_i changing and $\check{x}_i$ fixed; this is an explicit manifestation of the so-called Principle of Duality between lines and points of a projective plane—the fundamental property of a projective plane which implies that instead of viewing the points of the plane as the fundamental entities, and the lines as ranges (loci) of points, we may equally well take the lines as primary geometric entities and define points in terms of lines, characterizing a point by the complete set of lines passing through it. A quadratic equation

$$Q_{\check{x}\check{x}}(a) \equiv \sum_{i,j=1}^{3} a_{ij} \check{x}_i \check{x}_j = 0, \tag{2}$$

$a_{ij} = a_{ji}$ being real-valued constants, defines a (point-)conic. This conic can be proper (regular, det $a_{ij} \neq 0$) or degenerate (singular, det $a_{ij} = 0$); as for the former we distinguish between real and imaginary (i.e. having an empty image), while the latter comprises, as we will see in the subsequent

[1] A singular point of a planar curve is the point at which there exists no tangent to the curve.

sections, a pair of (real or complex-conjugated) lines, or a (double) real line (see, for example [9], p. 70). Taking any two distinct conics $Q_{\breve{x}\breve{x}}(a)=0$ and $Q_{\breve{x}\breve{x}}(b)=0$ we can define a unique, infinite set of conics called a (point-)pencil as

$$Q^{\vartheta}_{\breve{x}\breve{x}}(q)\equiv\sum_{i,j=1}^{3}q_{ij}(\vartheta_{1,2})\breve{x}_i\breve{x}_j\equiv\sum_{i,j=1}^{3}(\vartheta_1 a_{ij}+\vartheta_2 b_{ij})\breve{x}_i\breve{x}_j=0, \tag{3}$$

where the parameters $(\vartheta_1,\vartheta_2)\neq(0,0)$; from the last equation we see that the points shared by the two conics are common to all the conics of the pencil, and in what follows will be referred to as base points.

3. DEFINITIONS OF THE TEMPORAL AND SPATIAL

Intuition suggests that in order to make a successful move on the way towards the ultimate reconciliation of physical and psychological times, it seems to be necessary to treat time and space from the very beginning as standing on different footings.

This is the point of view implicitly adopted in both previous papers of ours [7, 8]. As already mentioned, the time coordinate was taken to be generated by a pencil of conics, while a spatial dimension was postulated to be induced by a pencil of lines, the carrier of the latter lying on a degenerate conic of the previous pencil. Speaking in more quantitative terms, a real proper projective conic (i.e. the element of an abstract temporal coordinate), being uniquely specified by five distinct points no three of which are collinear, needs, in general, six distinct quantities a_{ij}'s to be given, while only three different ζ_i's are sufficient to define a projective line (i.e. the element of an abstract spatial dimension), as the latter is unambiguously defined by (any) two distinct points lying on it.

This difference between the temporal and spatial becomes much more pronounced when these abstract (i.e. projective) notions are replaced by concrete (i.e. affine) ones, that is when the projective plane is affinized. Indeed, there are altogether three distinct types of real proper affine conics (i.e. ellipses, parabolas, and hyperbolae—this being precisely what enables us to define the arrow of time), which is in contrast with the only kind of real affine lines (the fact that accounts nicely for the observed ‘homogeneity’ of a spatial coordinate).

In order to give a more explicit realization of the above ideas let us take, as an example, the pencil of conics dealt with in [7]

$$Q^{\vartheta}_{\breve{x}\breve{x}}(q)\equiv\sum_{i,j=1}^{3}q_{ij}(\vartheta_{1,2})\breve{x}_i\breve{x}_j=\vartheta_1\breve{x}_1\breve{x}_2+\vartheta_2\breve{x}_3^2=0. \tag{4}$$

This pencil exhibits two base points ($\varrho\neq0$), B_1: $\varrho\breve{x}_i=(0,\ 1,\ 0)$ and B_2: $\varrho\breve{x}_i=(1,\ 0,\ 0)$, and two singular conics; $\vartheta(\equiv\vartheta_2/\vartheta_1)=\pm\infty$, i.e. a double real line $\breve{x}_3=0$, and $\vartheta=0$, i.e. a pair of real lines $\breve{x}_1=0$ and $\breve{x}_2=0$ whose point of intersection S: $\varrho\breve{x}_i=(0,0,1)$ is a singular point. These degenerates also separate the set of proper conics into two distinct families: $-\infty<\vartheta<0$ and $0<\vartheta<+\infty$.

Let us now affinize the above pencil by choosing the ideal line (dashed) in such a way that it incorporates neither of the base points $B_{1,2}$, nor the singular point S. It is obvious that the most general equation of a line meeting such constraints reads

$$\breve{x}_1-m\breve{x}_2-n\breve{x}_3=0 \tag{5}$$

if both m and n are non-zero (assumed, without any loss of generality, to be positive). Inserting this equation into eqn (4) yields

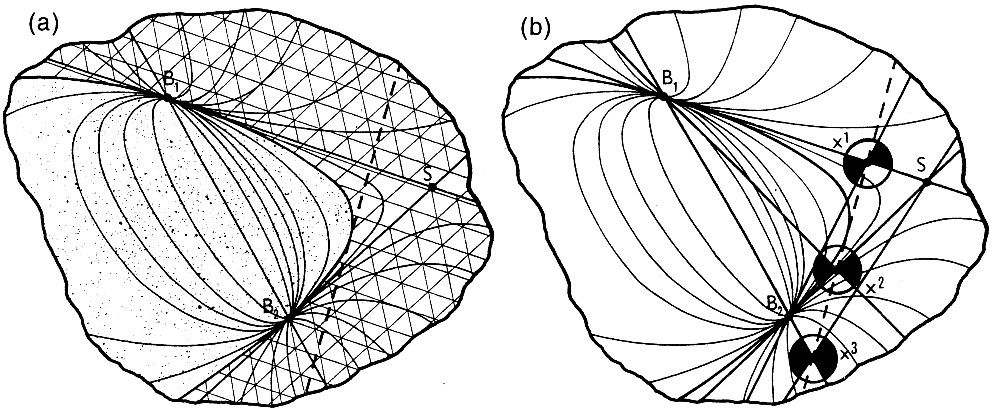

Fig. 1. An internal structure of the temporal dimension (a) and the multiplicity of spatial coordinates (b) of a space-time configuration generated by the 'regularly' affinized pencil of conics defined by eqn (4). The symbols and notation are explained in the text.

$$m\check{x}_2^2+n\check{x}_2\check{x}_3+\vartheta\check{x}_3^2=0, \tag{6}$$

which is easily recognized as a quadratic equation in the variable $\Delta\equiv\ \check{x}_2/\check{x}_3$, with the roots

$$\Delta_{1,2}=\frac{-n\pm\sqrt{n^2-4\vartheta m}}{2m}. \tag{7}$$

From the last expression it follows that whereas one family of proper conics, $-\infty<\vartheta<0$, consists, as illustrated in Fig. 1(a), solely of hyperbolae (thick shaded area), the other acquires a very interesting structurization featuring the domain of the hyperbolae ($\Delta_{1,2}$ distinct and real; shaded area)

$$0<\vartheta<n^2/4m \tag{8}$$

as well as the region of the ellipses ($\Delta_{1,2}$ distinct but complex; dotted area)

$$n^2/4m<\vartheta<+\infty, \tag{9}$$

the two being separated from each other by a single parabola ($\Delta_1\equiv\Delta_2$; the heavy-drawn curve)

$$\vartheta=n^2/4m. \tag{10}$$

And this is really a very remarkable pattern for it is seen to reproduce strikingly well, at least at a qualitative level, the observed arrow of time after we postulate [7] that "the events of the past (future) are represented by the hyperbolae (ellipses), and the parabola stands for a single moment of the present"; therefore, we shall henceforward call it the 'ordinary arrow', as we shall also call any other pattern showing qualitatively the same structurization, regardless of the character of the pencil of conics concerned.

In addition to being a mechanism that enables us, as we have just shown, to endow the temporal coordinate with a non-trivial internal structure, the above-described affinization of the projective plane also induces a very interesting sort of non-equivalence among the potential spatial dimensions, i.e. among the s-pencils. To see that we recall ([7]; see also the Introduction) our definition of an s-pencil as a pencil of lines whose carrier (i.e. the point shared by all the lines of the pencil) falls on (one of) the degenerate(s) of a given pencil of conics; in the particular case of pencil (4) it is thus any pencil of lines with the carrier lying on one of the lines SB_1, SB_2, or

B_1B_2. Since any affinization of a projective plane means singling out, or deleting from the latter one line, which is given a special status of the ideal line, or the 'line at infinity' (e.g. [9], p. 3), then it is quite natural to assume that those s-pencils which incorporate this line will also play a special physical role in our theory. Namely, we have stipulated [7, 8] that "only such pencils (henceforth referred to as s^*-pencils) generate the spatial dimensions we can observe". The puzzling three-dimensionality of the physical space is then laid bare as soon as we realise that the above 'regular' affinization of pencil (4) is characterized by just three distinct s^*-pencils. This is apparent from Fig. 1(b), where these pencils are represented by circles centered at corresponding carriers; we see, in particular, that two spatial coordinates (denoted as x^1 and x^2) are borne by the $\vartheta = 0$ degenerate, while the remaining one (x^3) is supported by the other, $\vartheta = \pm\infty$ singular conic.

4. ARROW OF TIME AND DIMENSIONALITY OF SPACE

As the next step let us have a look at what happens with the above-introduced notions of time and space if pencil (4) is affinized in a 'singular' way, i.e. when the ideal line passes via singular point S, or hits one of the base points, $B_{1,2}$.

The former case is clearly a limiting case of eqn (5) for $n \to 0$. Taking the corresponding limits of eqs. (8–10) we see that both the parabola and the domain of hyperbolae disappear; the whole family is thus composed of ellipses only and, so, generates no arrow. This change is accompanied by the $3 \to 2$ reduction in the number of spatial dimensions because the pencils representing the coordinates x^1 and x^2 (see Fig. 1(b)) become fused in this limit.

A completely analogous situation is encountered when the ideal line incorporates one of the base points, say B_2. This case is obtained in the limit when both $m \to \infty$ and $n \to \infty$, but $n/m \to d$, where d is a finite, non-zero real number. Now we see that it is the region of ellipses that dissolves together with the parabola, and the merging pencils, as it is obvious from Fig. 1(b), are those standing for the coordinates x^2 and x^3.

Even more pronounced transitions to arrow-free configurations are observed in the limit(s) when the ideal line coincides with the B_1B_2 line, or with one of lines SB_1, SB_2; as in these cases every point of the ideal line becomes the carrier of the s^*-pencil the corresponding observable space exhibits (uncountably) infinitely many dimensions.

From these findings and the results of the preceding section it follows that in the universe generated by pencil (4) there is a profound and intimate connection between the existence of temporal arrow and three-dimensionality of space, the two features being conditioned by each other. This is perhaps one of the most serious implications of our theory, whose importance will even become more evident after we show (Sections 7 and 8) that within the set of all possible types of conics' pencils in a real projective plane those of the type given by eqn (4) have special standing.

5. OTHER TEMPORAL ARROWS

We already mentioned in the introductory section that our approach also leads to temporal arrows that are structured differently from the above-described one. With the view of demonstrating that we will consider another pencil of conics, defined as

$$Q^{\vartheta}_{\check{x}\check{x}}(q) \equiv \sum_{i,j=1}^{3} q_{ij}(\vartheta_{1,2})\check{x}_i\check{x}_j = \vartheta_1\check{x}_1\check{x}_2 + \vartheta_2(\check{x}_2^2 - \check{x}_3^2) = 0. \quad (11)$$

It can easily be checked that this pencil is endowed with three base points

$$B_1: \varrho\check{x}_i = (1, 0, 0), \tag{12}$$

$$B_2: \varrho\check{x}_i = (0, 1, 1), \tag{13}$$

$$B_3: \varrho\check{x}_i = (0, -1, 1), \tag{14}$$

where ϱ is an arbitrary non-zero real number, and with two degenerate conics $\vartheta = 0$ and $\vartheta = \pm\infty$, either representing a pair of real lines, that is

$$\check{x}_1 = 0 \vee \check{x}_2 = 0 \tag{15}$$

and

$$\check{x}_2 + \check{x}_3 = 0 \vee \check{x}_2 - \check{x}_3 = 0, \tag{16}$$

and either possessing one singular point, i.e.

$$S_1: \check{x}_1 = \check{x}_2 = 0 \tag{17}$$

and

$$S_2: \check{x}_2 = \check{x}_3 = 0, \tag{18}$$

respectively; comparing eqn (12) and eqn (18) we see that

$$S_2 \equiv B_1. \tag{19}$$

Having briefly listed the most important projective properties of the pencil let us again affinize the projective plane and see what kind of 'stratification' this induces among the proper conics of the pencil. Following the strategy of the preceding section we will first handle the case when the ideal line avoids both the base and singular points. From eqs. (12–14, 17) and eqn (18) it follows that the most general equation of the line obeying this condition is still that given by eqn (5), but now supplemented by the constraints

$$m, n \neq 0 \wedge |m| \neq |n|. \tag{20}$$

After inserting eqn (5) into eqn (11) we arrive at

$$(\vartheta + m)\check{x}_2^2 + n\check{x}_2\check{x}_3 - \vartheta\check{x}_3^2 = 0 \tag{21}$$

which is solved by

$$\Delta_\pm \equiv \left(\frac{\check{x}_2}{\check{x}_3}\right)_\pm = \frac{-n \pm \sqrt{n^2 + 4\vartheta(\vartheta + m)}}{2(\vartheta + m)} = \frac{-n \pm 2\sqrt{(\vartheta - \vartheta_+)(\vartheta - \vartheta_-)}}{2(\vartheta + m)}, \tag{22}$$

where

$$\vartheta_\pm = (-m)\frac{1 \mp (|m|/m)\sqrt{1 - (n/m)^2}}{2}. \tag{23}$$

If we now look back at constraints (20) we see that two cases must be distinguished. The $|n| > |m|$ case is characterized by the complex (conjugated) $\vartheta_\pm$ and, so, by a purely homogeneous pencil consisting of hyperbolae. On the other hand, the $|n| < |m|$ mode, for which both ϑ_+ and ϑ_- are real, is characterized by a very intriguing structural pattern exhibiting two regions of hyperbolae, namely

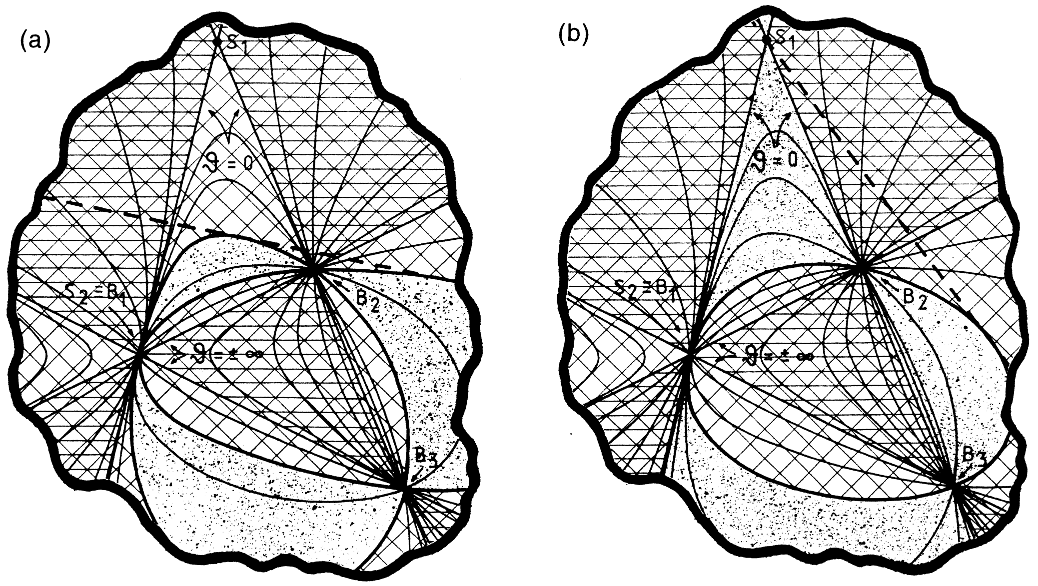

Fig. 2. The structure of a defective temporal dimension of the first kind (a) and of the ordinary arrow of time (b) characteristic for spacetime configurations generated by pencil (11). The symbols and notation are the same as in Fig. 1.

$$\vartheta > \vartheta_+ \wedge \vartheta > \vartheta_- \tag{24}$$

and

$$\vartheta < \vartheta_+ \wedge \vartheta < \vartheta_-, \tag{25}$$

but only a single domain of ellipses,

$$\mathrm{Min}\{\vartheta_+, \vartheta_-\} < \vartheta < \mathrm{Max}\{\vartheta_+, \vartheta_-\}, \tag{26}$$

the latter being separated from the former two domains by two distinct parabolas,

$$\vartheta = \vartheta_+ \tag{27}$$

and

$$\vartheta = \vartheta_-. \tag{28}$$

This is schematically depicted in Fig. 2(a) and rephrased in terms of physics it would correspond to "time endowed with two distinct domains of the past and two different moments of the present, but with only one region of the future!" This temporal structure is thus facing a sort of 'pathology' or a 'defect' for if having also two domains of the future it could perfectly be regarded as two-dimensional (like, e.g. that described in [8]). The rigorous analysis shows that it is really one dimensional only, and we will call it as a defective temporal dimension of the first kind.

In order to see its one-dimensionality we notice that the existence of two distinct degenerates, eqn (15) and eqn (16), implies that pencil (11), like the previous one, comprises just two separate families of real non-degenerate conics, one defined via

$$0 < \vartheta < +\infty \tag{29}$$

and the other through

$$-\infty < \vartheta < 0. \tag{30}$$

If we then look back at eqn (23) we find out that both the parabolas, eqn (27) and eqn (28), lie within one and the same family—that given by eqns (29) and (30) for $m<0$ ($m>0$). And thus, only one family is non-trivially stratified—the other shows a homogeneous pattern consisting solely of hyperbolae (see Fig. 2(a)).

As the next step, we will have a look at what happens if the second of inequalities (20) is relaxed, i.e. if we let

$$m \to \pm n. \tag{31}$$

From eqn (23) it then follows that

$$\vartheta_{\pm} \to \vartheta_0 \equiv -\frac{m}{2}, \tag{32}$$

which simply means that the two parabolas get merged together, causing the domain of the future to disappear completely; two sectors of the domain of the past are, however, preserved and given by (see eqn (24) and eqn (25))

$$\vartheta > \vartheta_0 \tag{33}$$

and

$$\vartheta < \vartheta_0. \tag{34}$$

The pseudo two-dimensional time of the previous paragraph is thus 'converted' into another peculiar type of "time that lacks the future, and where the 'past' is cut into two separate domains by a single moment of the present"—the distortion of the arrow we will call a defective temporal dimension of the second kind. From eqs. (13–14, 5) and eqn (31) it is obvious that this is always the case when the ideal line gets in touch with either B_2 or B_3.

It is quite surprising to see that these 'pathologic' temporal dimensions are generated by the affinizations whose ideal lines do not hit any of the singular points. Therefore, a question may naturally arise whether we can even find any ordinary (i.e. of the same type as found in Section 3) arrows within this pencil. As we will demonstrate below the answer to this question is positive.

To this end in view it is sufficient to relax now the first of inequalities (20), keeping $m \neq 0$ but letting

$$n \to 0 \tag{35}$$

in order to see that one of the parabolas

$$\vartheta = \vartheta_+ \to 0 \tag{36}$$

is deformed into a degenerate conic. Hence, we are left with only one parabola,

$$\vartheta = \vartheta_- \to -m, \tag{37}$$

i.e. with the single moment of the present. This also implies that in the limit the family endowed with a non-trivial structure gets rid of one domain of the past, because the region of the future is bounded from one side by the degenerate (36), and, as it follows from eqn (26), spans either

$$0 > \vartheta > -m \ (m>0) \tag{38}$$

or

$$-m > \vartheta > 0 \ (m<0). \tag{39}$$

The resulting configuration is schematically shown in Fig. 2(b). And thus, in spite of the fact that this affine image of pencil (11) is characterized by the ideal line passing through a singular point (S_1—see eqs. (17, 5) and eqn (35)) it generates a 'perfect' temporal arrow.

In order to gain further insight into the structure of different temporal arrows just discovered, and to establish also the context within which a proper understanding of these concepts can substantially develop and deepen, we will next examine the pencil

$$Q^{\vartheta}_{\check{x}\check{x}} \equiv \sum_{i,j=1}^{3} q_{ij}(\vartheta_{1,2})\check{x}_i\check{x}_j = -\vartheta_1\check{x}_1^2 + (\vartheta_1+\vartheta_2)\check{x}_2^2 - \vartheta_2\check{x}_3^2 = 0, \tag{40}$$

which is characterized by four base points ($\varrho \neq 0$)

$$B_1: \varrho\check{x}_i = (1, 1, 1), \tag{41}$$

$$B_2: \varrho\check{x}_i = (1, 1, -1), \tag{42}$$

$$B_3: \varrho\check{x}_i = (1, -1, 1), \tag{43}$$

$$B_4: \varrho\check{x}_i = (-1, 1, 1), \tag{44}$$

and has three degenerates, each comprising—as in the previous case—a pair of real lines

$$\vartheta = 0: \check{x}_1 = \check{x}_2 \vee \check{x}_1 = -\check{x}_2, \tag{45}$$

$$\vartheta = -1: \check{x}_1 = \check{x}_3 \vee \check{x}_1 = -\check{x}_3, \tag{46}$$

$$\vartheta = \pm\infty: \check{x}_2 = \check{x}_3 \vee \check{x}_2 = -\check{x}_3, \tag{47}$$

and containing one singular point

$$S_1: \check{x}_1 = \check{x}_2 = 0, \tag{48}$$

$$S_2: \check{x}_1 = \check{x}_3 = 0, \tag{49}$$

$$S_3: \check{x}_2 = \check{x}_3 = 0, \tag{50}$$

respectively. These degenerates also represent natural boundaries for the three separate families of real non-degenerate conics,

$$-\infty < \vartheta < -1, \tag{51}$$

$$-1 < \vartheta < 0, \tag{52}$$

and

$$0 < \vartheta < +\infty. \tag{53}$$

In order to see that pencil (40) generates temporal dimensions whose properties are almost identical with those induced by the previous pencil we will keep our discussion at the level that is as close as possible to the line of development followed in the preceding paragraphs of this section.

So, we again start with the case when the ideal line avoids both the singular and base points; the line is still defined via eqn (5), if the constraints read

$$m, n \neq 0 \wedge 1 \pm m \pm n \neq 0, \tag{54}$$

where all combinations of the signs are permitted. The combining of eqn (5) and eqn (40) yields

$$(\vartheta - m^2 + 1)\check{x}_2^2 - (\vartheta + n^2)\check{x}_3^2 - 2mn\check{x}_2\check{x}_3 = 0 \tag{55}$$

that is satisfied with

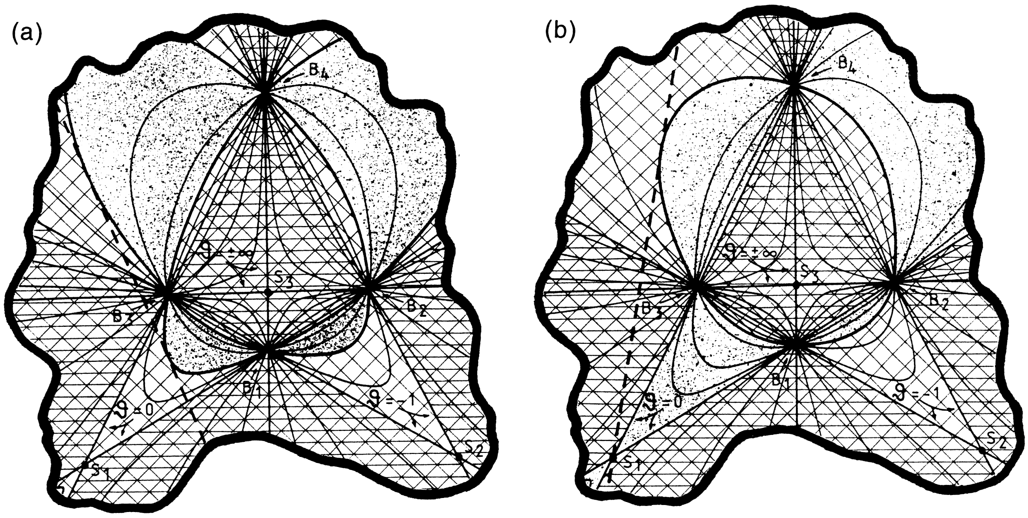

Fig. 3. The structure (a) of a defective temporal dimension of the first kind and (b) of the ordinary arrow of time, now induced by pencil (40). The symbols and notation are the same as in the previous figures.

$$\Delta_{\pm} \equiv \left(\frac{\check{x}_3}{\check{x}_2}\right)_{\pm} = -\frac{2mn \pm \sqrt{4m^2n^2 + 4(\vartheta + n^2)(\vartheta - m^2 + 1)}}{2(\vartheta + n^2)}$$

$$= -\frac{mn \pm \sqrt{(\vartheta - \vartheta_+)(\vartheta - \vartheta_-)}}{(\vartheta + n^2)}, \tag{56}$$

where now

$$\vartheta_{\pm} = \frac{m^2 - n^2 - 1}{2}\left(1 \pm \frac{|m^2 - n^2 - 1|}{m^2 - n^2 - 1}\sqrt{1 - \frac{4n^2}{(m^2 - n^2 - 1)^2}}\right). \tag{57}$$

Here we can again obtain either a purely homogeneous mode ($2|n| > |m^2 - n^2 - 1|$) consisting of hyperbolae, or a highly structured pattern ($2|n| < |m^2 - n^2 - 1|$) giving rise to a pathologic temporal dimension of the first kind, as clearly seen in Fig. 3(a) (compare with Fig. 2(a)). It lies with the reader to verify one-dimensionality of this coordinate, i.e. to prove that both the parabolas share, analogously to the previous case, one and the same family (e.g. that given by eqn (53) for $m^2 - n^2 - 1 > 0$).

The second kind of time pathology represents the transition between the two above described cases,

$$2n \rightarrow \pm(m^2 - n^2 - 1), \tag{58}$$

and again means the merging of the two parabolas

$$\vartheta_{\pm} \rightarrow \vartheta_0 \equiv \frac{m^2 - n^2 - 1}{2}, \tag{59}$$

and dissolution of the region of the future, while preserving the past being split into two segments. It is a quite straightforward exercise to verify that this is the case when the ideal line passes via any (but only one) of the base points.

As for the ordinary arrow of time this is produced if the ideal line incorporates one of the singular points, which for S_1 and S_2 is in the limit when, respectively,

$$\pm 1 \neq m \neq 0, n \rightarrow 0, \tag{60}$$

and

$$m \rightarrow 0, n \neq 0. \tag{61}$$

This in both cases makes one of the parabolas disappear, converting it into a degenerate, namely either

$$\vartheta = \vartheta_{-} \rightarrow 0, \tag{62}$$

or

$$\vartheta = \vartheta_{-} \rightarrow -1, \tag{63}$$

respectively, and at the same time forces out one of the two domains of the past so that the domain of future becomes bounded from one side by a degenerate—as shown in Fig. 3(b) for the case of the ideal line passing through S_1 (compare with Fig. 2(b)).

6. A COMPLETE CLASSIFICATION OF THE PENCIL-TEMPORAL

It turns out that the three (that is, ordinary and defective of the 1st and 2nd kinds) distinct types of one-dimensional temporal arrows, examined in the previous sections, together with the two-dimensional time described and analyzed in [8] exhaust all the possible kinds of temporal dimensions generated by the pencil of conics in a projective plane over the field of real numbers. This is quite obvious from Table 1 below, which gives a complete list of all projectivelly different sorts of real pencils of conics together with the information about whether or not (indicated by the 'plus' or 'minus' sign, respectively) a particular type of the temporal is generated by a given type of the pencil. Each pencil is characterized by the number of its base points, $\#_{BP}$, and by that of its singular conics, $\#_{SC}$, being denoted as $\#_{BP}\%\#_{SC}$; thus, for example, the pencils defined by eqs. (4, 11) and eqn (40) acquire in our new notation the symbols 2%2, 3%2, and 4%3,

Table 1. A synoptic classification of the temporal dimensions generated by the pencils of conics in a projective plane over the field of reals. The meaning of the individual symbols is given in the text

Type of conics' pencil	Type of temporal dimension				Note
	1-dimensional			2-dimensional	
	Ordinary	Defective			
		1st kind	2nd kind		
0%2	+	−	−	−	sd
0%3	+	−	−	+	
1%1	+	−	−	−	f, sd
1%2	+	−	−	+	
2%1A	−	+	+	−	f
2%1B	+	+	+	−	
2%2	+	−	−	−	f, sd
3%2	+	+	+	−	f
4%3	+	+	+	−	f

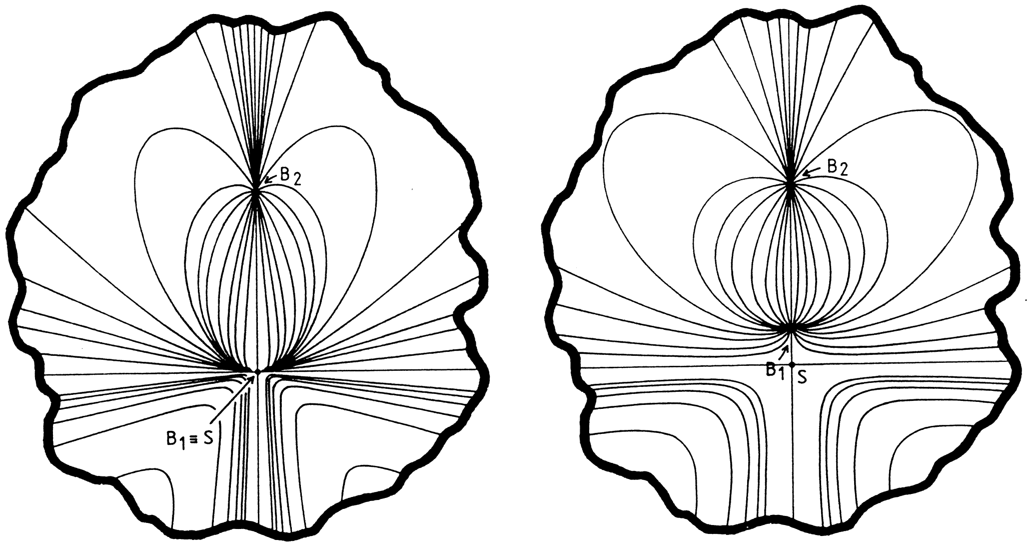

Fig. 4. A regular, i.e. corresponding to a selection of the affine line that avoids both the base ($B_{1,2}$) and singular (S) points, affine image of a pencil of conics of the projective type 2%1A (left) and 2%1B (right).

respectively, while the pencil of [8] is of the 0%3 type. It is worth mentioning that such classification is unambiguous except for the 2%1 type; here we must distinguish between two cases according to the fact whether one of the base points does (A) or does not (B) coincide with the singular point of the degenerate (the latter comprising a pair of real lines—see Fig. 4.)

Even a passing look at the table makes discernible a lot of interesting information. Thus, and perhaps not surprisingly, we find out that it is just the ordinary type of temporal arrow that is of the most frequent occurrence—generated by almost all types of pencils, the only exception being the 2%1A type (Figure 4, left). On the other hand, the two-dimensional times are seen to be rarest, being exhibited by only two sorts of pencils. We should also notice that the two kinds of defective temporal coordinates always emerge 'side by side', that is, there is no such (type of) pencil which would give rise to only one kind of temporal 'pathology'. However, the most important point of information for us is to see that there exist such pencils which induce ordinary arrows alone; this, as we will see soon, is so crucial a feature of our theory that it merits proper treatment in a separate section.

7. SELF-DUALITY AND ORDINARY ARROW(S) OF TIME

There are, as it follows from Tab. 1, three different types of pencils generating exclusively ordinary arrows, namely the 0%2, 1%1, and 2%2 types, and our next task is to find out what these have in common that distinguishes them from the remaining ones.

To this end in view we will return to eqn (2) of Section 2, which defines a conic in terms of point-coordinates. Slightly rephrased, a conic is viewed as a locus of points because it is the latter that we (have so far) regard(ed) as the fundamental constituents of the projective plane. In the same section it was, however, stressed that, according to the Principle of Duality, there is no reason to prefer (the coordinates of) points, $\breve{x}_i$, to (those of) lines, ζ_i. It means that a conic can equally well be defined in terms of the latter,

$$Q_{\zeta\zeta}(\alpha) \equiv \sum_{i,j=1}^{3} \alpha_{ij}\zeta_i\zeta_j = 0, \tag{64}$$

i.e. it can equivalently be looked upon as an envelope of lines; here $\alpha_{ij} = \alpha_{ji}$ are again real-valued constants, the conic itself being regarded, by a complete analogy with the point-case, as proper or degenerate in dependence on whether det α_{ij} deviates from zero or not. It is, of course, clear that eqn (2) and eqn (64) represent, in general, two distinct conics. In order for these equations to describe one and the same object the matrices a_{ij} and α_{ij} must be related to each other as follows (see, for example [10]; pp. 182–185)

$$\alpha_{ij} = \varrho a_{ij}^{-1}, (\varrho \neq 0) \tag{65}$$

with a_{ij}^{-1} standing for the inverse of the matrix a_{ij}; from the last equation it, *inter alia*, follows that the above-described dual view holds only for proper conics, as it is only a matrix with a non-zero determinant that can have the inverse.

Now let us consider two different proper conics and focus our attention on the pencil they define. At this point we have, however, at hand two equivalent definitions of the pencil; that is, in addition to the 'old', point definition represented by eqn (3), there is also a 'new', line definition that reads

$$Q_{\zeta\zeta}^{\lambda}(\kappa) \equiv \sum_{i,j=1}^{3} \kappa_{ij}(\lambda_{1,2})\zeta_i\zeta_j \equiv \sum_{i,j=1}^{3} (\lambda_1 a_{ij}^{-1} + \lambda_2 b_{ij}^{-1})\zeta_i\zeta_j = 0, \tag{66}$$

where $(\lambda_1, \lambda_2) \neq (0, 0)$. Despite the fact that both the definitions use one and the same couple of conics they, in general, do not lead to identical pencils; however, the interested reader can easily verify (having a look, if necessary, into e.g. [10, 11]) that if the point pencil is of a type that corresponds to one of the above-mentioned three types then it does look the same as its line counterpart, i.e. it is self-dual (being denoted by the symbol 'sd' in Tab. 1)! Hence, the one-dimensional temporal dimensions endowed with ordinary arrows are a unique and natural implication of our theory as soon as the requirement of self-duality is imposed on the character of pencils of conics. Or, rephrased in a layman-friendly language, the fact that there seem to be no other alternatives for the structure of the arrow of time to that we are familiar with from our everyday experience can simply reflect Nature's fondness for symmetry.

8. FULLNESS AS ANOTHER SYMMETRY CONSTRAINT

So, imposing a self-duality constraint upon our theory makes it possible to considerably reduce (9→3) the number of the types of pencils which can be regarded as relevant when attacking the fundamental question why the observed world has just such properties as it has, and not other. It is, however, interesting to notice that there exists still another important property belonging to the realm of conics' pencils that, if required to be met by the 'relevant' pencils, enables us to further reduce the above-mentioned number—the property that is referred to as the 'fullness' (and symbolized by a letter 'f' in Tab. 1) of a pencil (see, e.g. [11], p. 192 ff). Let us, therefore, spare some more lines by dedicating ourselves to examine this feature in some detail.

Let us again consider a general (point-)pencil of conics, $Q(q)$, and focus our attention on the number and character of the base points it possesses. To this purpose, we select two distinct conics $Q(a)$ and $Q(b)$ of the pencil, taking the latter to be proper, and choose the coordinate system in which the equation of this conic acquires the canonical form

Table 2. The five distinct categories of the full point-pencils of conics in a projective plane over reals. For the explanation of the symbols see the text

Category	N	μ	Type
V	1	one quadruple root	1%1
IV	2	one simple, one triple root	2%1A
III	2	two double roots	2%2
II	3	two simple roots, one double root	3%2
I	4	four simple roots	4%3

$$Q_{\check{x}\check{x}}(b) \equiv \sum_{i,j=1}^{3} b_{ij}\check{x}_i\check{x}_j = \check{x}_1\check{x}_3 - \check{x}_2^2 = 0. \quad (67)$$

Supposing that no base point lies on the $\check{x}_1 = 0$ line we can eliminate $\check{x}_3$ in the above equation and after substituting that into eqn (2) we have ($\check{x} \equiv \check{x}_2/\check{x}_1$)

$$a_{33}\check{x}^4 + 2a_{23}\check{x}^3 + (2a_{13} + a_{22})\check{x}^2 + 2a_{12}\check{x} + a_{11} = 0. \quad (68)$$

Since each solution of this equation specifies one base point, and distinct solutions correspond to distinct base points, our task thus converts into finding the total number (N) and multiplicity (μ) of the roots of the last equation. Here, we must distinguish between the already-mentioned full pencils, i.e. the pencils for which the polynomial on the r.h.s. of eqn (68) can be completely factored into (not necessarily distinct) linear factors (this being always the case when the ground field is algebraically closed), and the non-full pencils, where such a factorization is impossible. Speaking in quantitative terms, a given pencil is full iff

$$\Psi \equiv \sum_{k=1}^{N} \mu_k = 4, \quad (69)$$

from where it follows that there are altogether five distinct categories of such pencils, as listed in Tab. 2 (for more details, see [10], p. 255 ff).

There are thus only two different types of conics' pencils which meet the condition of being both full and self-dual; the already analyzed 2%2 type (Sections 3 and 4) and the 1%1 one. Although these two types of pencils of conics, being self-dual, both exhibit ordinary temporal arrows, they do differ as for the number of spatial dimensions generated.

9. DIMENSIONALITY OF SPACE REVISITED

In order to be more explicit we will take and analyze a particular pencil of the 1%1 type,

$$Q_{\check{x}\check{x}}^{\vartheta} \equiv \sum_{i,j=1}^{3} q_{ij}(\vartheta_{1,2})\check{x}_i\check{x}_j = \vartheta_1(\check{x}_1^2 + \check{x}_2^2 - \check{x}_3^2) + \vartheta_2(\check{x}_2 - \check{x}_3)^2 = 0. \quad (70)$$

Obviously, its single degenerate corresponds to $\vartheta = \pm\infty$, being a (double) real line $\check{x}_2 = \check{x}_3$, and the only base point B has the coordinates $\varrho\check{x}_i = (0, 1, 1)$. Since we are again interested in the affine structure of the pencil we insert eqn (5) into eqn (70) to obtain

$$(m^2 + \vartheta + 1)\check{x}_2^2 + 2(mn - \vartheta)\check{x}_2\check{x}_3 + (n^2 + \vartheta - 1)\check{x}_3^2 = 0, \quad (71)$$

the last equation being clearly met by

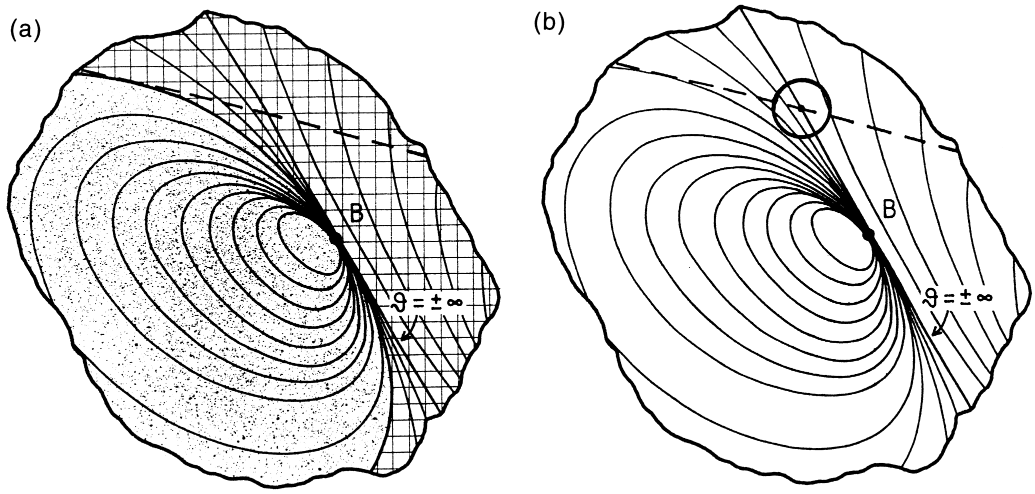

Fig. 5. The ordinary temporal arrow (a) and the single spatial coordinate (b) of a space-time configuration generated by the regularly affinized pencil of conics of the 1%1 type, eqn (70). The symbols and notation are identical with those of Fig. 1.

$$\Delta_{\pm} \equiv \left(\frac{\breve{x}_2}{\breve{x}_3}\right)_{\pm} = \frac{\vartheta - mn \pm \sqrt{(\vartheta - mn)^2 - (\vartheta + m^2 + 1)(\vartheta + n^2 - 1)}}{\vartheta + m^2 + 1} =$$

$$= \frac{\vartheta - mn \pm \sqrt{(\vartheta_0 - \vartheta)(m+n)^2}}{\vartheta + m^2 + 1}, \tag{72}$$

with

$$\vartheta_0 \equiv \frac{m^2 - n^2 + 1}{(m+n)^2}. \tag{73}$$

Focusing on regular affinizations, i.e. ones characterized by the ideal line not passing via the base point B, we must demand $m \neq -n$, which means (see eqn (73)) that the conic corresponding to ϑ_0 is proper and, after being affinized, becomes thus a parabola (the single moment of the present). Equation (72) further tells us that this parabola bounds, from one side, the region of hyperbolae (the past, $\vartheta < \vartheta_0$) and, from the other, the domain of ellipses (the future, $\vartheta > \vartheta_0$)—a typical ordinary arrow, as visualized in Fig. 5(a).

Although this arrow is qualitatively identical to that induced by pencils of the 2%2 type (Section 3, Fig. 1(a)), the two are apparently different when their quantitative structure is considered. And it is just the dimensionality of the corresponding spatial configurations when this difference shows up in a most conspicuous way: whereas the space coupled to a 2%2 arrow is, as shown in Section 3 (Figure 1(b)), three-dimensional, a 1%1 arrow is accompanied, as depicted in Fig. 5(b), by a single spatial coordinate. And thus, although we might have serious problems to distinguish by experiment whether we live in the universe generated by type 1%1 pencils or in that induced by type 2%2 ones when taking into account only their temporal aspects, such potential troubles will completely disappear as soon as we turn to their spatial characteristics. At this stage of the theory development we have, however, no knowledge of any other (symmetry) principle underlying nature's preference for the type 2%2 pencils.

10. CONCLUSION

We have further developed and studied the concept of pencil-generated space-times. As shown in great detail, the concept turns out to be a remarkably fertile framework not only for grasping successfully the qualitative aspect of the observed arrow of time, but also for showing a profound link of the latter with another fundamental puzzle faced by physics, the multiplicity of spatial coordinates. Although it must be admitted that all of the above is, by this stage, quite an abstract theory, the alert reader will surely not need strong convincing of another very important message this article conveys to us: our universe might also have been endowed with a differently structured temporal dimension (Sections 5 and 6) than the one we all are familiar with from our everyday experience (Section 3). The possible explanation why this is not the case (Section 7) stimulates a host of conceptual questions, all very difficult to answer, no doubt, but all definitely within the domain of physics. What seems to be, however, beyond any doubt here is the fact that the above-outlined formalism might prove to be crucial for tackling successfully (perhaps) one of the most fundamental questions of mankind, namely "Where lies the physics of mind?" [1].

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