International Journal of Geometric Methods in Modern Physics
Vol. 8, No. 1 (2011) 39–47

DOI: 10.1142/S0219887811004951

ON THE VELDKAMP SPACE OF GQ(4, 2)

METOD SANIGA
Astronomical Institute, Slovak Academy of Sciences
SK-05960 Tatranská Lomnica, Slovak Republic
msaniga@astro.sk

Received 21 January 2010
Accepted 20 June 2010

The Veldkamp space, in the sense of Buekenhout and Cohen, of the generalized quadrangle GQ(4, 2) is shown not to be a (partial) linear space by simply giving several examples of Veldkamp lines (V-lines) having two or even three Veldkamp points (V-points) in common. Alongside the ordinary V-lines of size five, one also finds V-lines of cardinality three and two. There, however, exists a subspace of the Veldkamp space isomorphic to PG(3, 4) having 45 perps and 40 plane ovoids as its 85 V-points, with its 357 V-lines being of four distinct types. A V-line of the first type consists of five perps on a common line (altogether 27 of them), the second type features three perps and two ovoids sharing a tricentric triad (240 members), whilst the third and fourth type each comprises a perp and four ovoids in the rosette centered at the (common) center of the perp (90). It is also pointed out that 160 non-plane ovoids (tripods) fall into two distinct orbits — of sizes 40 and 120 — with respect to the stabilizer group of a copy of GQ(2, 2); a tripod of the first/second orbit sharing with the GQ(2, 2) a tricentric/unicentric triad, respectively.

Keywords: GQ(4, 2); geometric hyperplane; Veldkamp space; PG(3, 4).

1. Introduction

Generalized quadrangles of types GQ(2, t), with $t = 1, 2$ and 4, have recently been found to play a prominent role in quantum information and black hole physics; the first type for grasping the geometrical nature of the so-called Mermin squares [1, 2], the second for underlying commutation properties between the elements of two-qubit Pauli group [1–3], and the third one for fully encoding the $E_{6(6)}$ symmetric entropy formula describing black holes and black strings in $D = 5$ [4]. Whereas GQ(2, 2) is isomorphic to its point-line dual, this is not the case with the remaining two geometries; the dual of GQ(2, 1) being GQ(1, 2), that of GQ(2, 4) GQ(4, 2) [5]. These two duals, strangely, did not appear in the above-mentioned physical contexts. It is, therefore, natural to ask why this is so. We shall try to shed light on this matter by invoking the concept of the Veldkamp space of a point-line incidence structure [6]. The Veldkamp space of GQ(2, 4) was shown to be a linear space isomorphic to PG(5, 2) [7]. Here, we shall demonstrate that the Veldkamp space of GQ(4, 2), due to the existence of two different kinds of ovoids in GQ(4, 2), is not even a partial linear space, though it contains a (linear) subspace isomorphic to PG(3, 4) in which GQ(4, 2) lives as a non-degenerate Hermitian variety.

2. Rudiments of the Theory of Finite Generalized Quadrangles and Veldkamp Spaces

To make our exposition as self-contained as possible, and for the reader's convenience as well, we will first gather all essential information about finite generalized quadrangles [5], then introduce the concept of a geometric hyperplane [8] and, finally, that of the Veldkamp space of a point-line incidence geometry [6].

A *finite generalized quadrangle* of order (s,t), usually denoted GQ(s,t), is an incidence structure $S = (P, B, \mathrm{I})$, where P and B are disjoint (non-empty) sets of objects, called respectively points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms [5]: (i) each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line; (ii) each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point and, (iii) if x is a point and L is a line not incident with x, then there exists a unique pair $(y, M) \in P \times B$ for which $x\mathrm{I}M\mathrm{I}y\mathrm{I}L$, from these axioms it readily follows that $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$. It is obvious that there exists a point-line duality with respect to which each of the axioms is self-dual. Interchanging points and lines in S thus yields a generalized quadrangle S^D of order (t,s), called the dual of S. If $s = t$, S is said to have order s. The generalized quadrangle of order $(s,1)$ is called a grid and that of order $(1,t)$ a dual grid. A generalized quadrangle with both $s > 1$ and $t > 1$ is called thick. Every finite generalized quadrangle is obviously a *partial* linear space, that is, the point-line incidence structure where (a) any line has at least two points, and (b) any two points are on at most one line.

Given two points x and y of S one writes $x \sim y$ and says that x and y are collinear if there exists a line L of S incident with both. For any $x \in P$ define the perp of x as $x^{\perp} = \{y \in P \,|\, y \sim x\}$ and note that $x \in x^{\perp}$, being its center, obviously, $|x^{\perp}| = 1 + s + st$. Given an arbitrary subset A of P, the *perp* of A, $A^{\perp}$, is defined as $A^{\perp} = \bigcap\{x^{\perp} \,|\, x \in A\}$ and $A^{\perp\perp} := (A^{\perp})^{\perp}$; in particular, if x and y are two non-collinear points, then $\{x,y\}^{\perp\perp}$ is called a hyperbolic line (through them). A triple of pairwise non-collinear points of S is called a *triad*; given any triad T, a point of $T^{\perp}$ is called its center and we say that T is acentric, centric or unicentric according as $|T^{\perp}|$ is, respectively, zero, non-zero or one. An ovoid of a generalized quadrangle S is a set of points of S such that each line of S is incident with exactly one point of the set; hence, each ovoid contains $st + 1$ points. The dual concept is that of spread; this is a set of lines such that every point of S is on a unique line of the spread. A *rosette* of ovoids is a set of ovoids through a given point x of S partitioning the set of points non-collinear with x. A *fan* of ovoids is a set of ovoids partitioning the whole point set of S; if S has order (s,t) then every rosette contains s ovoids and every fan features $s+1$ ovoids.

A *geometric hyperplane* H of a point-line geometry $\Gamma(P,B)$ is a proper subset of P such that each line of Γ meets H in one or all points [8]. For $\Gamma = \mathrm{GQ}(s,t)$, it is well known that H is one of the following three kinds [5]: (i) the perp of a point x, $x^{\perp}$; (ii) a (full) subquadrangle of order (s,t'), $t' < t$; and (iii) an ovoid.

Finally, we shall introduce the notion of the *Veldkamp space* of a point-line incidence geometry $\Gamma(P,B)$, $\mathcal{V}(\Gamma)$ [6]. $\mathcal{V}(\Gamma)$ is the space in which: (i) a point is a geometric hyperplane of Γ; and (ii) a line is the collection H_1H_2 of all geometric hyperplanes H of Γ such that $H_1 \cap H_2 = H_1 \cap H = H_2 \cap H$ or $H = H_i$ $(i = 1,2)$, where H_1 and H_2 are distinct points of $\mathcal{V}(\Gamma)$.

3. Basic Properties of GQ(4, 2)

The unique generalized quadrangle $\mathrm{GQ}(4,2)$, associated with the classical group $\mathrm{PGU}_4(2)$, can be represented by 45 points and 27 lines of a non-degenerate Hermitian surface $H(3,4)$ in $\mathrm{PG}(3,4)$, the three-dimensional projective space over $\mathrm{GF}(4)$ [5, 9, 10]. Every line has five points and there are three lines through every point. This quadrangle features both unicentric and tricentric triads,[a] and has no spreads. There are 16 tricentric triads through each point; hence, their total number is $45 \times \frac{16}{3} = 240$. Its geometric hyperplanes are (45) perps of points and (200) ovoids, because it has no subquadrangles of type $\mathrm{GQ}(4,1)$ [5].

Obviously, perps correspond to the cuts of $H(3,4)$ by its 45 tangent planes. As first shown by Brouwer and Wilbrink [9], ovoids fall into two distinct orbits of sizes 40 and 160. The ovoids of the first orbit are called *plane* ovoids, as each of them represents a section of $H(3,4)$ by one of the 40 non-tangent planes. The ovoids of the second orbit are referred to as *tripods*, each comprising nine isotropic points on three hyperbolic lines $\{x,y\}^{\perp\perp}$, $\{x,z\}^{\perp\perp}$ and $\{x,w\}^{\perp\perp}$, where $\{x,y,z,w\}$ is a basis of non-isotropic points. Given a plane ovoid $\mathcal{P}$ and any two distinct points $x, y \in \mathcal{P}$, it is *always* true that $\{x,y\}^{\perp\perp} \subseteq \mathcal{P}$. Hence, $\{\mathcal{P} \backslash \{x,y\}^{\perp\perp}\} \cup \{x,y\}^{\perp}$ is again an ovoid, and all the tripods can be obtained in this manner from plane ovoids. $\mathrm{GQ}(4,2)$ contains both fans and rosettes of ovoids [9]. There are altogether 520 fans, each made of a plane ovoid and four tripods and falling into two orbits of sizes 480 and 40, and 26 rosettes on a given point; two of them feature four plane ovoids, the remaining ones consist of four tripods each.

Apart from perps and ovoids, $\mathrm{GQ}(4,2)$ is endowed with one more kind of distinguished subgeometry — that isomorphic to the unique generalized quadrangle $\mathrm{GQ}(2,2)$; this is, however, not a geometric hyperplane. There are altogether 36 distinct copies of $\mathrm{GQ}(2,2)$ living inside $\mathrm{GQ}(4,2)$, and with any of them an ovoid is found to share a triad. For a plane ovoid this triad is always tricentric. For tripods,

[a] $\mathrm{GQ}(4,2)$ is also endowed with acentric triads, but these are of no relevance for our subsequent reasoning.

however, it can be either tricentric (40 of them) or unicentric (120 of them); in what follows we shall occasionally refer to the former/latter as tri-tripods/uni-tripods, respectively. Two plane ovoids overlap in either a single point or a (tricentric) triad. Given a plane ovoid, there are 10 tri-tripods and 30 uni-tripods disjoint from it. There exists a remarkable partitioning of the point-set of GQ$(4,2)$ in terms of three GQ$(2,1)$s and three GQ$(1, 2)$s such that one of the latter group forms with each of the former group a GQ$(2,2)$. Another noteworthy property is the existence of pairs of plane ovoids and/or tri-tripods on the common (tricentric) triad whose symmetric difference is a disjoint union of two GQ$(1,2)$s.

All the above-mentioned properties can be ascertained — some readily, some requiring a bit of work — from a diagrammatical illustration of GQ$(4,2)$ depicted in Fig. 1. In this picture, of a form showing an automorphism of order five, all the 45 points of GQ$(4,2)$ are represented by bullets, whereas its 27 lines have as many as four distinct representations: two are represented by (concentric) circles, five by arcs of parabolas touching the inner circle, another five by parabolas touching the outer circle, five by arcs of ellipses, and, finally, 10 by straight line-segments. (Note that there are many intersections of segments and arcs that do not stand for any point of GQ$(4,2)$.) A copy of GQ$(2,2)$ is also highlighted (black bullets, all line-segments and all arcs of ellipses).

4. Distinguished Features of the Veldkamp Space of GQ$(4,2)$

4.1. *Linear subspace isomorphic to* PG$(3,4)$

In PG$(3,4)$ a point and a plane are duals of each other. On the other hand, both a perp and a plane ovoid are associated each with a unique plane of PG$(3,4)$. Hence,

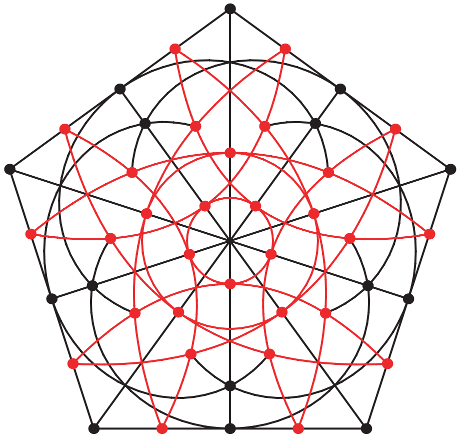

Fig. 1. A diagrammatical model of the structure of GQ$(4,2)$ whose points are illustrated by bullets and lines by straight segments, arcs of ellipses and/or parabolas, and two circles (for more details, see the text). Note a particular copy of GQ$(2,2)$ (black), its complement (gray) being nothing but famous Schläfli's double-six of lines.

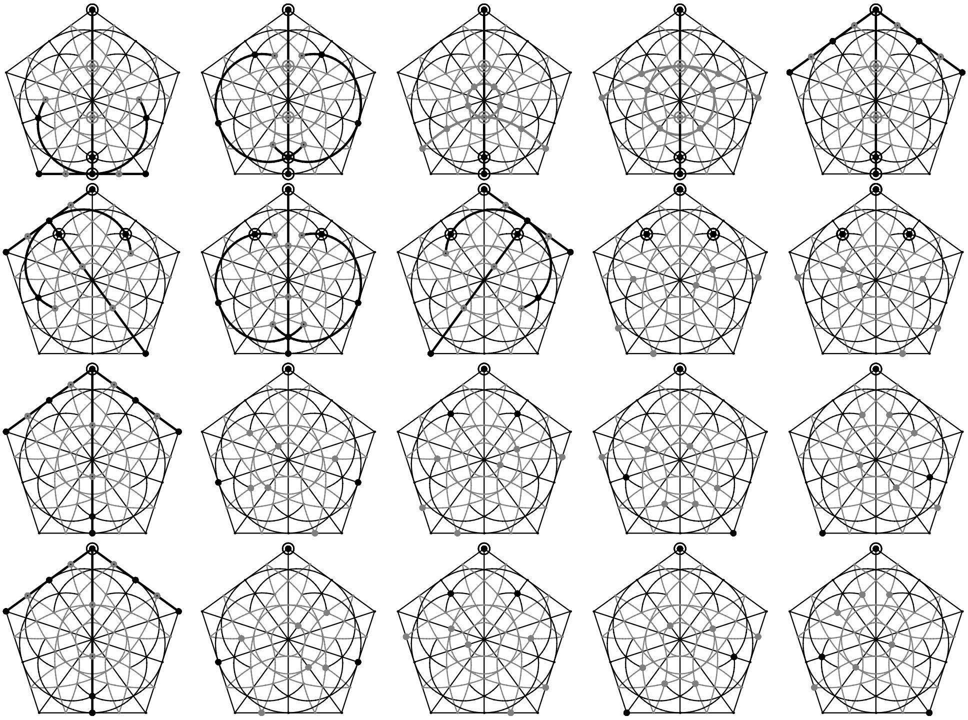

Fig. 2. Representatives of the four different types of V-lines forming the linear subspace of $\mathcal{V}(\mathrm{GQ}(4,2))$ isomorphic to $\mathrm{PG}(3,4)$. The encircled bullets represent the points shared by all the five V-points forming a given V-line.

disregarding tripods for the moment, we find a subspace of the Veldkamp space of $\mathrm{GQ}(4,2)$ that is isomorphic to $\mathrm{PG}(3,4)$. The 85 V-points of this subspace are 45 perps and 40 planar ovoids, and the 357 V-lines split into four distinct types as shown in Fig. 2. A V-line of the first type (1-st row in Fig. 2) consists of five perps on a common pentad of collinear points i.e. on a common line; clearly, there are 27 V-lines of this type as each line leads to a unique V-line. A second-type V-line (2-nd row) features three perps and two ovoids sharing a tricentric triad; since each such triad defines a unique V-line, there are altogether 240 V-lines of this type. Third-/fourth-type V-lines (3-rd/4-th row) each comprises a perp and four ovoids in the rosette centered at the perp's center (the only common point); their total number thus amounts to $2 \times 45 = 90$.

4.2. *Examples of V-lines on three and two common points*

In order to show that $\mathcal{V}(\mathrm{GQ}(4,2))$ is not a (partial) linear space it suffices to find two (or more) V-lines sharing two (or more) V-points. From the previous subsection it should already be fairly obvious that it is the existence of tripods that prevents $\mathcal{V}(\mathrm{GQ}(4,2))$ from being a linear space.

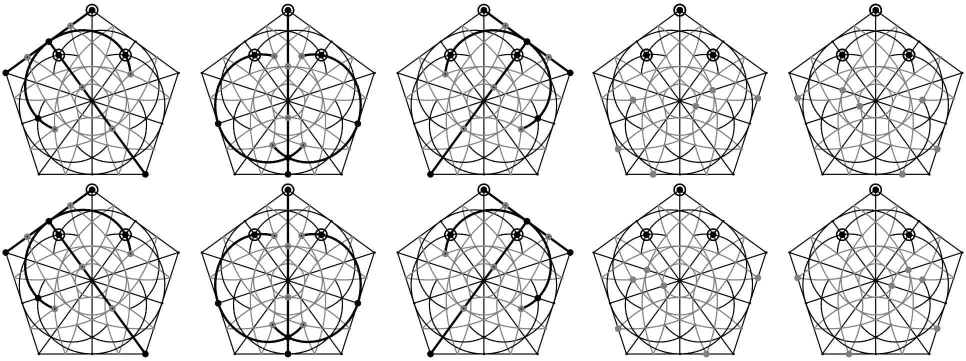

Fig. 3. An instance of two distinct V-lines having three V-points in common. We note in passing that the tripods are tri-tripods with respect to the selected copy of GQ$(2, 2)$.

To this end, let us again have a look at a type-two V-line of the PG$(3, 4)$-subspace, reproduced once again in Fig. 3, top row. It turns out that we can get a new type of V-line by replacing its two plane ovoids by two particular tripods, as shown in Fig. 3, bottom row. Hence, we have an example of two distinct V-lines having three V-points (the three perps) in common.

The next couple of examples feature three (Fig. 4) and four (Fig. 5) V-lines through two common V-points. In the first case, the V-lines consist each of a perp

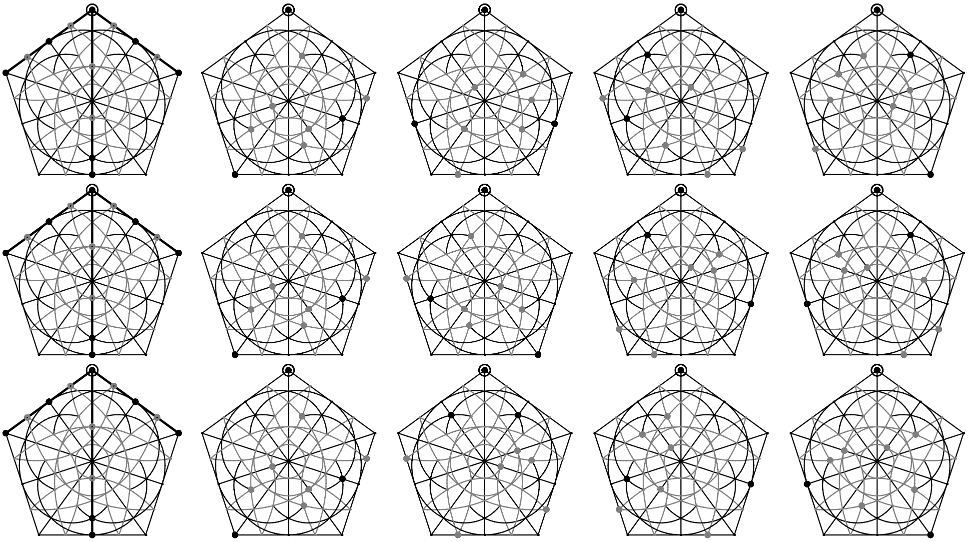

Fig. 4. An example of three different V-lines on two common V-points (a perp and a tri-tripod).

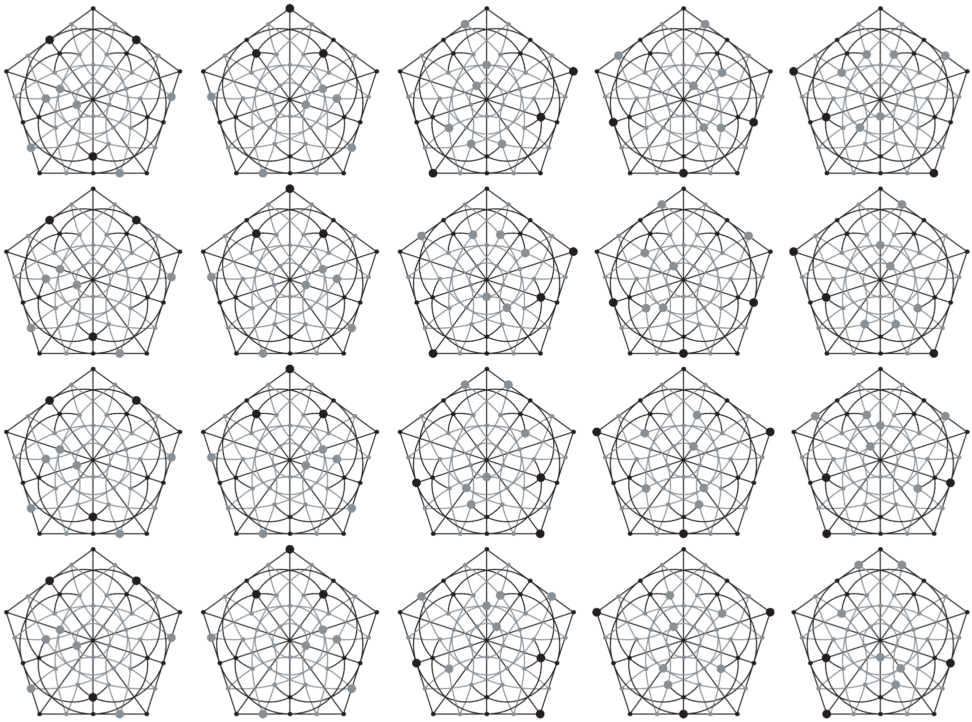

Fig. 5. An example of four different V-lines sharing two V-points (a plane ovoid and a tri-tripod).

and four tripods in the *rosette* centered at the perp's center; the perp and the first tripod are the common V-points. With respect to the selected GQ$(2, 2)$, the first two tripods in each V-line are tri-tripods, the other two being uni-tripods. In the second case, each of the four V-lines represents a *fan* of ovoids, the first ovoid being planar, second a tri-tripod, and the remaining three uni-tripods.

4.3. *Examples of V-lines of sizes three and two*

That the structure of $\mathcal{V}(\mathrm{GQ}(4, 2))$ is much more complex and intricate than that of any (partial) linear space is, alongside the above-introduced examples, also illustrated by the existence of V-lines of cardinality less than five. Thus, we found many V-lines of size three, like the one shown in Fig. 6. This V-line consists of a perp and two uni-tripods on a common unicentric triad; since two perps, obviously, cannot share a unicentric triad and a given perp contains 48 such triads, we find altogether $45 \times 48 = 2160$ V-lines of this particular kind. Finally, there are also a large number of V-lines of size two; the one depicted in Fig. 7 is composed of a plane ovoid and a tripod having six points in common.

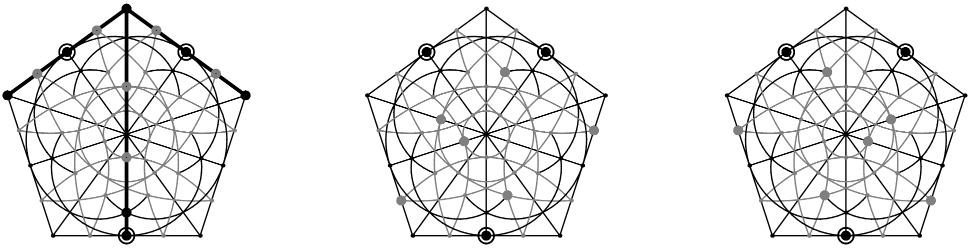

Fig. 6. An example of V-line of size three.

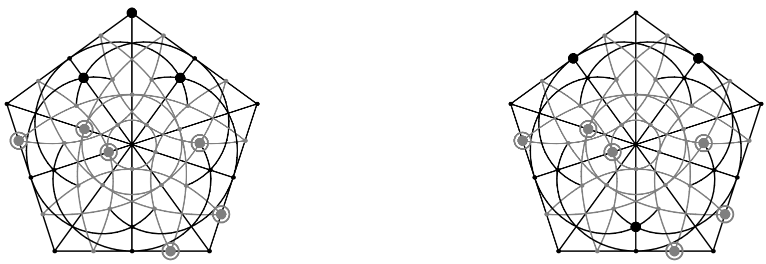

Fig. 7. An example of V-line of size two.

5. Conclusion

We have furnished several examples showing that the Veldkamp space of GQ$(4,2)$ is not a (partial) linear space. This is in a sharp contrast with the case of the dual, GQ$(2,4)$, whose Veldkamp space is a linear space, being isomorphic to PG$(5,2)$ [7]. We surmise that this fact, among other things, might also contain an important clue why it is GQ$(2,4)$ and not GQ$(4,2)$, which is relevant for a particular black-hole/qubit correspondence [4].

Acknowledgments

This work was conceived within the framework of the Cooperation Group "Finite Projective Ring Geometries: An Intriguing Emerging Link Between Quantum Information Theory, Black-Hole Physics, and Chemistry of Coupling" at the Center for Interdisciplinary Research (ZiF), University of Bielefeld, Germany, being completed with the support of the VEGA Grant agency, projects Nos. 2/0092/09, 2/0098/10 and 2/7012/27. I am extremely grateful to Prof. Hans Havlicek (Vienna University of Technology) for providing me with an easy-to-use software for generating all the figures. I also thank Richard Green (University of Colorado, Boulder) for motivating combinatorial remarks.

References

[1] M. Planat and M. Saniga, On the Pauli graphs of N-qubits, *Quant. Inf. Comput.* **8** (2008) 127–146 (arXiv: quant-ph/0701211).

[2] M. Saniga, M. Planat and P. Pracna, Projective ring line encompassing two-qubits, *Theoret. Math. Phys.* **155** (2008) 905–913 (arXiv: quant-ph/0611063).

[3] H. Havlicek, B. Odehnal and M. Saniga, Factor-group-generated polar spaces and (multi-)qubits, *SIGMA* **5** (2009) 096 (arXiv: 0903.5418).

[4] P. Lévay, M. Saniga, P. Vrana and P. Pracna, Black hole entropy and finite geometry, *Phys. Rev. D* **79** (2009) 084036 (arXiv: 0903.0541).

[5] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles* (Pitman, Boston–London–Melbourne, 1984) see also K. Thas, *Symmetry in Finite Generalized Quadrangles* (Birkhäuser, Basel, 2004).

[6] F. Buekenhout and A. M. Cohen, *Diagram Geometry* (Chapter 10.2). (Springer, Berlin, 2009). Preprints of separate chapters can be also found at http://www.win.tue.nl/~amc/buek/.

[7] M. Saniga, R. M. Green, P. Lévay, P. Pracna and P. Vrana, The Veldkamp space of GQ(2, 4), *Int. J. Geom. Meth. Mod. Phys.* **7** (2010) 1133–1145, submitted (arXiv: 0903.0715).

[8] M. A. Ronan, Embeddings and hyperplanes of discrete geometries, *European J. Combin.* **8** (1987) 179–185.

[9] A. E. Brouwer and H. A. Wilbrink, Ovoids and fans in the generalized quadrangle Q(4, 2), *Geom. Dedicata* **36** (1990) 121–124.

[10] J. W. P. Hirschfeld, On the history of generalized quadrangles, *Bull. Belg. Math. Soc.* **3** (1994) 417–421.