

Geometric Hyperplanes of the Near Hexagon $L_3 \times GQ(2, 2)$

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Abstract. Having in mind their potential quantum physical applications, we classify all geometric hyperplanes of the near hexagon that is a direct product of a line of size three and the generalized quadrangle of order two. There are eight different kinds of them, totalling to $1,023 = 2^{10} - 1 = |\text{PG}(9, 2)|$, and they form two distinct families intricately related with the points and lines of the Veldkamp space of the quadrangle in question.

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1. Introduction

There are quite a few finite geometries/point-line incidence structures that have recently been recognized to play an important role in physics. Amongst them, the one that acquired a particular footing is $GQ(2, 2)$ —the unique generalized quadrangle of order two. On its own, the $GQ(2, 2)$ is the underlying framework for fully expressing the commutation relations between the elements of a two-qubit generalized Pauli group in geometrical terms [1, 2]. As a subgeometry/subconfiguration, the $GQ(2, 2)$ underpins a particular kind of truncation of the $E_{6(6)}$ -symmetric black hole/black string entropy formula in five dimensions [3]. In both the cases, remarkably, it is also the geometric hyperplanes of the $GQ(2, 2)$ that enter the game in an essential way.

In view of these developments, it is likely that there are other physically relevant finite geometries incorporating GQ(2, 2)s. In this paper we shall have a look at the most promising candidate of them: the slim dense near hexagon that originates as a direct product of a projective line over the field of two elements and a GQ(2, 2), together with the totality of its geometric hyperplanes.

2. Generalized Quadrangles, Near Polygons, Geometric Hyperplanes and Veldkamp Spaces

We start with a brief overview of the essential theory and nomenclature; for more details, the interested reader is referred to [4–6].

A *finite generalized quadrangle* of order (s, t) , usually denoted GQ(s, t), is an incidence structure $S = (P, B, I)$, where P and B are disjoint (non-empty) sets of objects, called respectively points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms [4]: (i) each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line; (ii) each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point; and (iii) if x is a point and L is a line not incident with x , then there exists a unique pair $(y, M) \in P \times B$ for which $xIMyIL$; from these axioms it readily follows that $|P| = (s + 1)(st + 1)$ and $|B| = (t + 1)(st + 1)$. It is obvious that there exists a point-line duality with respect to which each of the axioms is self-dual. If $s = t$, S is said to have order s . The generalized quadrangle of order $(s, 1)$ is called a grid and that of order $(1, t)$ a dual grid. A generalized quadrangle with both $s > 1$ and $t > 1$ is called thick. Given two points x and y of S one writes $x \sim y$ and says that x and y are collinear if there exists a line L of S incident with both. For any $x \in P$ denote $x^\perp = \{y \in P | y \sim x\}$ and note that $x \in x^\perp$; obviously, $|x^\perp| = 1 + s + st$. Given an arbitrary subset A of P , the *perp(-set)* of A , A^\perp , is defined as $A^\perp = \bigcap \{x^\perp | x \in A\}$. A triple of pairwise non-collinear points of S is called a *triad*; given any triad T , a point of T^\perp is called its center and we say that T is *acentric*, *centric* or *unicentric* according as $|T^\perp|$ is, respectively, zero, non-zero or one. An *ovoid* of a generalized quadrangle S is a set of points of S such that each line of S is incident with exactly one point of the set; hence, each ovoid contains $st + 1$ points.

A *near polygon* (see, e.g., [5] and references therein) is a connected partial linear space $S = (P, B, I)$, $I \subset P \times L$, with the property that given a point x and a line L , there always exists a unique point on L nearest to x . (Here distances are measured in the point graph, or collinearity graph of the geometry.) If the maximal distance between two points of S is equal to d , then the near polygon is called a near $2d$ -gon. A near 0-gon is a point and a near 2-gon is a line; the class of near quadrangles coincides with the class of generalized quadrangles. A non-empty set X of points in a near polygon $S = (P, B, I)$ is called a *subspace* if every line meeting X in at least two points is completely contained in X . A subspace X is called *geodetically closed* if every point on a shortest path between two points of X is

contained in X . Given a subspace X , one can define a sub-geometry S_X of S by considering only those points and lines of S which are completely contained in X . If X is geodetically closed, then S_X clearly is a sub-near-polygon of S . If a geodetically closed sub-near-polygon S_X is a non-degenerate generalized quadrangle, then X (and often also S_X) is called a *quad*.

A near polygon is said to have order (s, t) if every line is incident with precisely $s + 1$ points and if every point is on precisely $t + 1$ lines. If $s = t$, then the near polygon is said to have order s . A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance two have at least two common neighbors. A near polygon is called *slim* if every line is incident with precisely three points. It is well known (see, e.g., [4]) that there are, up to isomorphism, three slim non-degenerate generalized quadrangles. The (3×3) -grid is the unique generalized quadrangle $GQ(2, 1)$. The unique generalized quadrangle $GQ(2, 2)$, often dubbed the doily, is the generalized quadrangle of the points and those lines of $PG(3, 2)$ which are totally isotropic with respect to a given symplectic polarity. The points and lines lying on a given non-singular elliptic quadric of $PG(5, 2)$ define the unique generalized quadrangle $GQ(2, 4)$. Any *slim dense* near polygon contains quads, which are necessarily isomorphic to either $GQ(2, 1)$, $GQ(2, 2)$ or $GQ(2, 4)$.

The incidence structure S is called the direct product of S_1 and S_2 , and denoted by $S_1 \times S_2 (\simeq S_2 \times S_1)$, if: (i) $P := P_1 \times P_2$; (ii) $B := (P_1 \times B_2) \cup (B_1 \times P_2)$; and (iii) the point (x, y) of S is incident with the line $(z, L) \in P_1 \times B_2$ if and only if $x = z$ and yI_2L and with the line $(M, w) \in B_1 \times P_2$ if and only if xI_1M and $y = w$.

Next, a *geometric hyperplane* of a partial linear space is a proper subspace meeting each line (necessarily in a unique point or the whole line). For $S = GQ(s, t)$, it is well known that H is one of the following three kinds: (i) the perp-set of a point x , x^\perp ; (ii) a (full) subquadrangle of order (s, t') , $t' < t$; and (iii) an ovoid. The set of points at non-maximal distance from a given point x of a dense near polygon S is a hyperplane of S , usually called the *singular* hyperplane with *deepest* point x . Given a hyperplane H of S , one defines the *order* of any of its points as the number of lines through the point which are fully contained in H ; a point of a hyperplane/sub-configuration is called *deep* if all the lines passing through it are fully contained in the hyperplane/sub-configuration. If H is a hyperplane of a dense near polygon S and if Q is a quad of S , then precisely one of the following possibilities occurs: (1) $Q \subseteq H$; (2) $Q \cap H = x^\perp \cap Q$ for some point x of Q ; (3) $Q \cap H$ is a sub-quadrangle of Q ; and (4) $Q \cap H$ is an ovoid of Q . If case (1), case (2), case (3), or case (4) occurs, then Q is called, respectively, *deep*, *singular*, *sub-quadrangular*, or *ovoidal* with respect to H .

Finally, we shall introduce the notion of the *Veldkamp space* of a point-line incidence geometry $S(P, B, I)$, $\mathcal{V}(S)$ [6], which is the space in which (i) a point is a geometric hyperplane of S and (ii) a line is the collection $H'H''$ of all geometric hyperplanes H of S such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where H' and H'' are distinct points of $\mathcal{V}(S)$.

3. $L_3 \times GQ(2, 2)$ and its Geometric Hyperplanes

3.1. $L_3 \times GQ(2, 2)$

The unique point-line incidence geometry $L_3 \times GQ(2, 2)$ is obtained by taking three isomorphic copies of the generalized quadrangle $GQ(2, 2)$ and joining the corresponding points to form lines of size 3. It is a slim dense near hexagon having 45 points and 60 lines, with four lines through a point. The number of common neighbors of two points x, y at distance two, that is $|x^\perp \cap y^\perp|$, is either two or three. $L_3 \times GQ(2, 2)$ contains 15 $GQ(2, 1)$ -quads (henceforth simply grid-quads) and three $GQ(2, 2)$ -quads (doily-quads). The lines of $L_3 \times GQ(2, 2)$ are of two distinct types according as they lie in three grid-quads (type one) or a grid-quad and a doily-quad (type two); there are 15 lines of type one and 45 of type two, with each point being on one line of type one and three lines of type two. Also, each grid-quad features both kinds of lines in equal proportion, whereas a doily-quad consists solely of type-two lines. This near hexagon can be universally embedded in $PG(9, 2)$ and its full group of automorphisms is isomorphic to $S_6 \times S_3$ [7, 8]. The structure of $L_3 \times GQ(2, 2)$ is fully encoded in the properties of its geometric hyperplanes, which we will now focus on.

3.2. THE VELDKAMP SPACE OF $GQ(2, 2)$

To this end, we shall first recall basic properties of the Veldkamp space of the doily, $\mathcal{V}(GQ(2, 2)) \simeq PG(4, 2)$, whose in-depth description can be found in [9]. The 31 points of $\mathcal{V}(GQ(2, 2))$, that is the 31 distinct copies of geometric hyperplanes of $GQ(2, 2)$, are of three distinct types: 15 perp-sets, 10 grids and five ovoids – as illustrated in Figure 1. The 155 lines of $\mathcal{V}(GQ(2, 2))$, each being of the form $\{H', H'', \overline{H'\Delta H''}\}$ where H' and H'' are two distinct geometric hyperplanes and $\overline{H'\Delta H''}$ is the complement of their symmetric difference, split into five distinct types as summarized in Table I and depicted in Figure 2.

3.3. GEOMETRIC HYPERPLANES OF $L_3 \times GQ(2, 2)$

Employing a “cubic pentagon” pictorial representation of $L_3 \times GQ(2, 2)$ as shown in Figure 3, it was quite a straightforward task to find out all the types of geometric hyperplanes of this geometry and to ascertain their basic characteristics, as summarized in Table II. There are eight different kinds of them and they form two distinct families according as they contain a deep doily-quad (H_1 to H_3) or not (H_4 to H_8). The fine structure of the hyperplanes of the first family is given in Figure 4, that of the second family in Figure 5. Comparing Figure 4 with Figure 1 and Figure 5 with Figure 2 one readily recognizes that this two-family split has a natural explanation in terms of the points and lines of $\mathcal{V}(GQ(2, 2))$.

A hyperplane of the first family is always of such form that the two doily-quads which are not deep must not only contain the hyperplanes of the same type, but

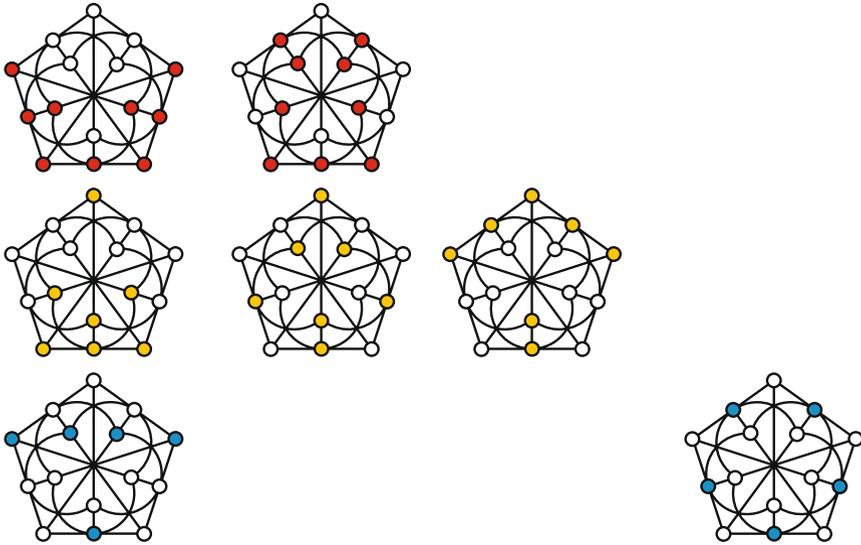


Figure 1. The three kinds of geometric hyperplanes of $GQ(2,2)$. The points of the quadrangle are represented by *small circles* and its *lines* are illustrated by the straight segments as well as by the segments of circles; note that not every intersection of two segments counts for a point of the quadrangle. The upper panel shows perp-sets (*yellow bullets*), the middle panel grids (*red bullets*) and the bottom panel ovoids (*blue bullets*). Each picture—except that in the bottom right-hand corner—stands for five different hyperplanes, the four other being obtained from it by its successive rotations through 72° around the center of the pentagon (Color online only).

Table I. A succinct summary of the properties of the five different types of the lines of $\mathcal{V}(GQ(2,2))$ in terms of the core (i.e., the set of points common to all the three hyperplanes forming a line) and the types of geometric hyperplanes featured by a generic line of a given type

Type	Core	Perps	Ovoids	Grids	No.
I	Pentad	1	0	2	45
II	Collinear triple	3	0	0	15
III	Tricentric triad	3	0	0	20
IV	Unicentric Triad	1	1	1	60
V	Single point	1	2	0	15

The last column gives the total number of lines per each type

these must be joined by the same type-one lines. Since each of the three doily-quads can be deep, this implies $|H_1|=3\times$ (the number of grids in $GQ(2,2)$) $=30$, $|H_2|=3\times$ (the number of perp-sets in $GQ(2,2)$) $=45$ and $|H_3|=3\times$ (the number of ovoids in $GQ(2,2)$) $=18$; altogether, $|H_1|+|H_2|+|H_3|=3\times$ (the number of Veldkamp *points* in the $\mathcal{V}(GQ(2,2))$) $=93$. On the other hand, there is an obvious one-to-one correspondence between the five types of hyperplanes of the second family and the five types of Veldkamp lines of $\mathcal{V}(GQ(2,2))$.

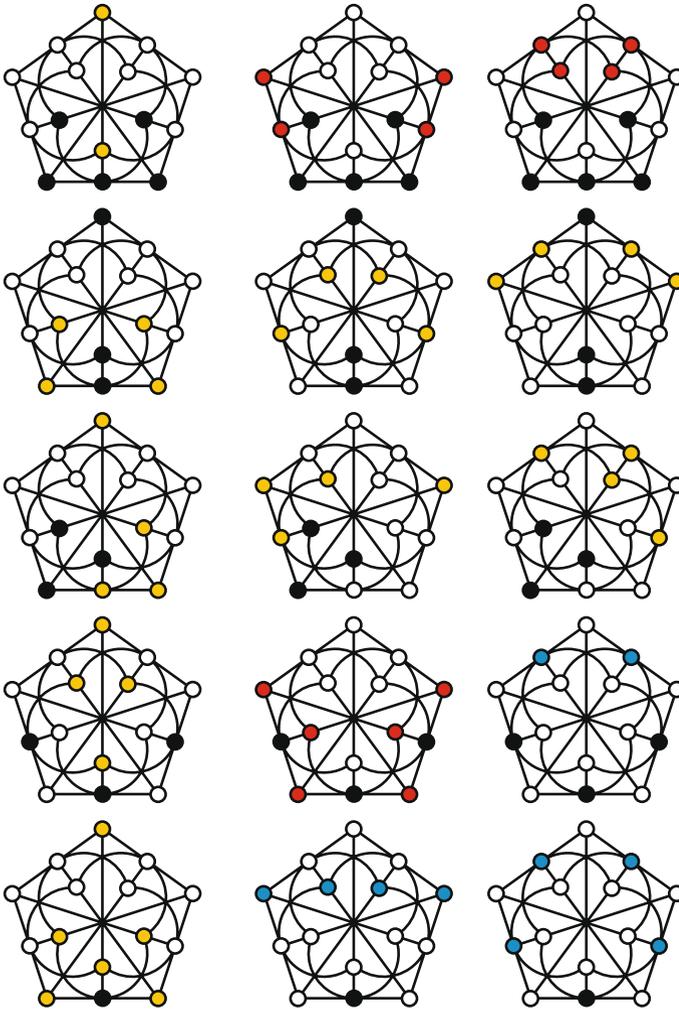


Figure 2. The five different kinds of the lines of $\mathcal{V}(\text{GQ}(2,2))$, each being uniquely determined by the properties of its core (black bullets) (Color online only).

In the following, let H be a hyperplane that contains no deep doily-quad. If the intersection of H with two of the three doilies is given, the intersection of H with the third is immediate. We use this to construct the remaining five types of geometric hyperplanes of $L_3 \times \text{GQ}(2,2)$. It is straightforward to check supposing the contrary, that every hyperplane has a singular doily. So, let one doily be singular with deep point P . If a second doily is singular, too, with deep point Q , there arise two different kinds of hyperplanes according to whether the distance of P and Q is 2 or 3. If it is 2, the third doily is singular, too, with deep point the third point of the unique triad in the grid-quad through P and Q ; this is type H_5 . The number of type H_5 hyperplanes is $15 \times 6 = 90$. If it is 3 and P' and Q' are the points

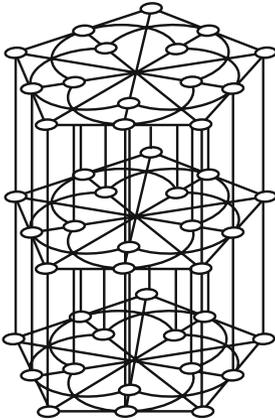


Figure 3. A pictorial representation of $L_3 \times GQ(2, 2)$. In this representation the three doily-quads lie in three parallel planes and the 15 lines of type one, “tying” them together and lying fully in grid-quads, “penetrate” these planes perpendicularly.

Table II. An overview of the types of geometric hyperplanes of the near hexagon $L_3 \times GQ(2, 2)$

Tp	Pt	Ln	No. of points of order					No. of grid-quads				No. of doily-quads				Cd
			0	1	2	3	4	dp	sg	ov	sq	dp	sg	ov	sq	
H_1	33	36	0	0	0	24	9	6	9	0	–	1	0	0	2	30
H_2	29	28	0	0	12	8	9	3	12	0	–	1	2	0	0	45
H_3	25	20	0	10	0	10	5	0	15	0	–	1	0	2	0	18
H_4	25	20	0	2	12	10	1	2	9	4	–	0	1	0	2	270
H_5	21	12	0	12	6	0	3	1	6	8	–	0	3	0	0	90
H_6	21	12	0	9	9	3	0	0	9	6	–	0	3	0	0	120
H_7	21	12	2	6	9	4	0	0	9	6	–	0	1	1	1	360
H_8	17	4	8	8	0	0	1	0	3	12	–	0	1	2	0	90

For each type (Tp) of a hyperplane we give the number of points (Pt) and lines (Ln), followed by the cardinalities of the points of a given order, cardinalities of deep (dp), singular (sg), ovoidal (ov) and subquadrangular (sq) quads of both kinds, and, finally, the total number of its copies (Cd)

of the third doily collinear with P , respectively Q , then no grid-quad is deep, the third doily is singular with deep point the third point of one of the triads through P' and Q' ; this is type H_6 and it holds $|H_6| = 15 \times 8 = 120$. Now suppose a second doily is subquadrangular. If the point P' of the subquadrangular doily collinear with P belongs to the subquadrangle, the third doily is subquadrangular, too; this is type H_4 which consists of $3 \times 15 \times 3 \times 2 = 270$ hyperplanes where 3 possibilities for the singular quad with 15 different deep points exist, where three choices of subquadrangles through P' exist and where two choices remain for subquadrangles in the third doily. If P' does not belong to the subquadrangle, the third

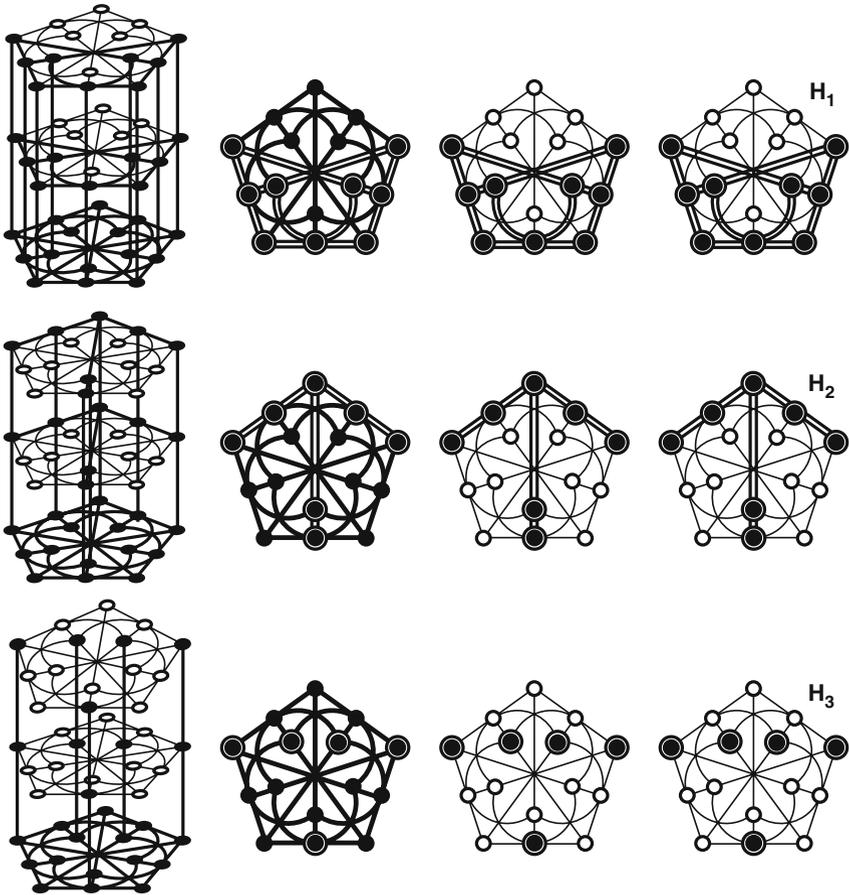


Figure 4. A diagrammatic illustration of the composition of the three kinds of geometric hyperplanes of the first family (see Table II). In each row, the first picture of the four shows a compact, cubic pentagonal view of the hyperplane; to avoid its too crowded appearance, only those type-one lines are drawn that belong to the hyperplane in question. In an alternative view furnished by the remaining three pictures, instead of being stacked on top of each other, the doily-quads are put side by side to make finer traits of the structure more discernible. The points and lines of a hyperplane are **boldfaced**. If the point is **encircled**, then the type-one line passing through it is fully contained in the hyperplane; the *doubled (type-two) lines* are those which belong to a deep grid-quad. Note that singular hyperplanes are those of type H_2 .

doily is ovoidal; this is type H_7 with $3 \times 10 \times 2 \times 6 = 360$ members where there are 3 choices for the subquadrangular doily with 10 possibilities for the choice of the grid, then 2 choices for the singular doily with 6 choices for the deep point of its singular hyperplane. If a second doily is ovoidal, we are back in type H_7 if the deep point of the singular quad is not collinear with any point of the ovoid of the ovoidal doily. If it is collinear with a point of the ovoid, then the third doily is ovoidal, too; this is type H_8 with 3 choices for the two ovoidal doilies, 6 choices

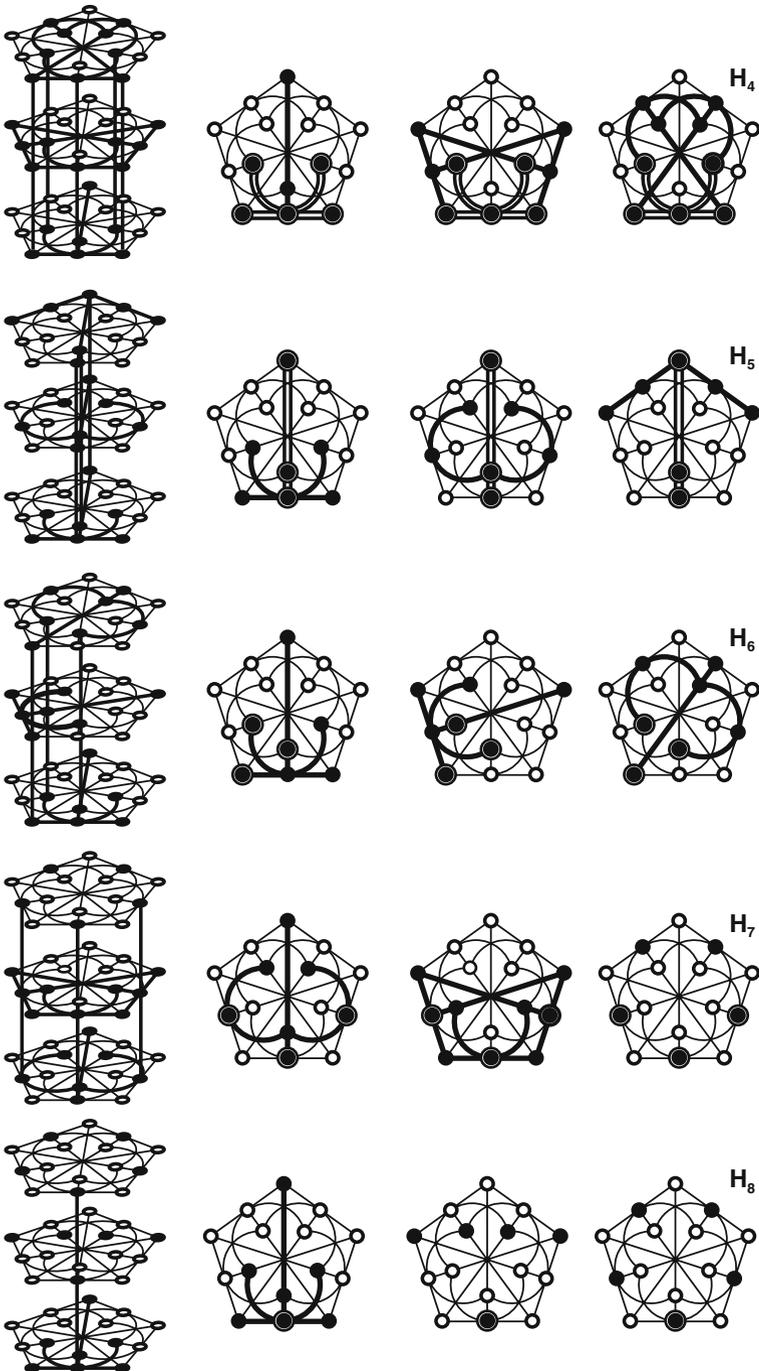


Figure 5. A diagrammatic illustration of the structure of the five kinds of geometric hyperplanes of the second family with the same symbols/notation as in Figure 4.

for the ovoid in one of the two ovoidal doilies and 5 possibilities for the deep point of the singular doily, altogether $3 \times 6 \times 5 = 90$. All in all, $|H_4| + |H_5| + \dots + |H_8| = 6 \times (\text{the number of Veldkamp lines in the } \mathcal{V}(\text{GQ}(2, 2))) = 930$.

The cardinalities of hyperplanes thus sum up to $93 + 930 = 1,023 = 2^{10} - 1 = |\text{PG}(9, 2)|$. This also proves that we have enumerated all hyperplanes since any slim dense near hexagon is universally embeddable into a projective space over $\text{GF}(2)$ and the geometric hyperplanes are precisely those arising from its universal embedding [8], and that in our particular case – as already mentioned in Sec. 3.1 – the universal embedding is indeed into $\text{PG}(9, 2)$ [7].

Table II reveals a number of interesting facts. First, one readily observes that $|H_i| \equiv 1 \pmod{4}$ for any i as per their *point* cardinality.¹ Next, there is no hyperplane featuring points of every order. Similarly, there is no hyperplane endowed with all the four kinds of doily-quads. On the other hand, there are two distinct types of hyperplane (H_4 and H_5) containing all the three kinds of grid-quads and every single hyperplane contains a singular grid-quad. Interestingly, there are a couple of types of hyperplane devoid of deep points (H_6 and H_7) and the same number of those having isolated points (that is, points of order zero – H_7 and H_8). Furthermore, there are as many as three distinct line types having identical point/line cardinality (H_5 to H_7). It is also worth mentioning that the complement of an H_1 is a pair of dual grids, $\text{GQ}(1, 2)_s$ – a distant relative of Schläfli's double-six.

We conclude this section with the following observation. Picking up a grid in one of the three doilies, there is a unique associated slim dense near hexagon of type $L_3 \times \text{GQ}(2, 1) \simeq L_3^{\times 3}$ sitting inside $L_3 \times \text{GQ}(2, 2)$. This (smallest slim dense) near hexagon features five distinct kinds of geometric hyperplanes [11] and viewing it as embedded in $L_3 \times \text{GQ}(2, 2)$, it can easily be demonstrated that each of them arises from a hyperplane of $L_3 \times \text{GQ}(2, 2)$; in fact, one can prove a stricter condition, namely that every hyperplane of $L_3 \times \text{GQ}(2, 1)$ originates from one of the hyperplanes of type H_2 , H_5 or H_6 , because the totality of these span a subspace of $\text{PG}(9, 2)$ isomorphic to $\text{PG}(7, 2)$, and so to $\mathcal{V}(L_3^{\times 3})$ as well (see [11]). We also note in passing that since two distinct copies of a grid in a doily always meet in a pair of concurrent lines (see Figure 2, top row), two different $L_3 \times \text{GQ}(2, 1)$'s of $L_3 \times \text{GQ}(2, 2)$ share a pair of concurrent grid-quads.

4. Conclusion

We have given a detailed description of all different types of geometric hyperplanes of the point-line incidence geometry $L_3 \times \text{GQ}(2, 2)$, which is the smallest

¹In this respect our near hexagon resembles the *dual* of the split Cayley hexagon of order two, where $|H_i| \equiv 3 \pmod{4}$, any i [10].

slim dense near hexagon featuring two distinct kinds of quads – namely grid-quads and doily-quads. The hyperplanes, whose total number amounts to $1,023 = 2^{10} - 1 = |\text{PG}(9, 2)|$, are of eight types and they form two distinct families according as they contain a deep doily-quad or not. This two-family split was demonstrated to have a natural explanation in terms of the points and lines of the Veldkamp space of the generalized quadrangle of order two. Each hyperplane’s type is uniquely characterized by the following string of parameters (Table II): the number of points and lines of a representative, followed by the cardinalities of the points of a given order, cardinalities of deep, singular, ovoidal and subquadrangular quads of both kinds, and, finally, by the total number of its copies. Several interesting combinatorial properties were also explicitly mentioned.

We believe that $L_3 \times \text{GQ}(2, 2)$, like $\text{GQ}(2, 2)$ itself, will play a prominent role in the context of both quantum information theory and entropy formulas of some yet unknown stringy black hole/ring solutions. It is especially the former domain when we surmise that the combinatorics and geometry of $L_3 \times \text{GQ}(2, 2)$ could mimic a whole class of *three two-qubit* systems entangled in a particular way and lead to the notion of a “twisted” Mermin square. In this respect, a particularly attractive task to address is as follows. It is a well-established fact [1,2] that the structure of $\text{GQ}(2, 2)$ underlies the commutation relations between the 15 operators of two-qubit Pauli group, and that grids sitting in it generate Mermin magic squares. Now, suppose that we label all the three doilies of $L_3 \times \text{GQ}(2, 2)$ by two-qubit Pauli matrices. We start from the configuration in which the labels on each line of type one are the same. Then we keep labels of one of the doilies fixed and let the group S_6 act on the labels/points of the other two. An interesting question emerges: how many Mermin squares can we get among *grid-quads* and how are they coupled to each other? (In our starting position there are none.) $L_3 \times \text{GQ}(2, 2)$ may even turn out to be of relevance for three-qubits as its automorphism group, $S_6 \times S_3$, is isomorphic to a maximal subgroup of $W'(E_7)$ that can be given an “entangled” three-qubit representation [12]. Explorations along these lines are already well under way and will be dealt with in a separate paper.

From a mathematical point of view, as the very next step it is desirable to find the stabilizer group for a representative of each hyperplane’s type and the corresponding point orbit sizes. Then, we shall embark on examining all the types of Veldkamp lines of $\mathcal{V}(L_3 \times \text{GQ}(2, 2))$; since, obviously, $\mathcal{V}(L_3 \times \text{GQ}(2, 2)) \simeq \text{PG}(9, 2)$, this will be a much more demanding task because $\text{PG}(9, 2)$ is endowed with as many as 174251 lines.

Finally, it is well worth mentioning that geometric hyperplanes have already been classified for several other slim dense near hexagons (some of which contain $\text{GQ}(2, 2)$ -quads), see [13,14] for the dual polar space $\text{DW}(5, 2)$, [15,16] for the dual polar space $\text{DH}(5, 4)$, [17] for the M_{24} near hexagon, [18] for the $U_4(3)$ near hexagon and [19] for the near hexagon on the 2-factors of the complete graph K_8 . These should also play some role in the above-outlined physical context.

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References

1. Saniga, M., Planat, M., Pracna, P.: Projective ring line encompassing two-qubits. *Theor. Math. Phys.* **155**(3), 905–913 (2008). arXiv:quant-ph/0611063
2. Planat, M., Saniga, M.: On the Pauli graphs of N -qudits. *Quantum Inf. Comput.* **8**(1–2), 127–146 (2008). arXiv:quant-ph/0701211
3. Lévy, P., Saniga, M., Vrana, P., Pracna, P.: Black hole entropy and finite geometry. *Phys. Rev. D* **79**(8), 084036 (2009). arXiv:0903.0541
4. Payne, S.E., Thas, J.A.: *Finite Generalized Quadrangles*. Pitman, Boston (1984)
5. De Bruyn, B.: *Near Polygons*. Birkhäuser, Basel (2006)
6. Buekenhout, F., Cohen, A.M.: *Diagram Geometry*. Springer, Berlin (2009). Preprints of separate chapters can be also found at <http://www.win.tue.nl/~amc/buek/>
7. Brouwer, A.E., Cohen, A.M., Hall, J.I., Wilbrink, H.A.: Near polygons and Fischer spaces. *Geom. Dedicata* **49**, 349–368 (1994)
8. Ronan, M.A.: Embeddings and hyperplanes of discrete geometries. *Eur. J. Combin.* **8**, 179–185 (1987)
9. Saniga, M., Planat, M., Pracna, P., Havlicek, H.: The Veldkamp space of two-qubits. *SIGMA* **3**, 075 (2007). arXiv:0704.0495
10. Frohardt, D., Johnson, P.M.: Geometric hyperplanes in generalized hexagons of order $(2, 2)$. *Comm. Algebra* **22**, 773–797 (1994)
11. Green, R.M., Saniga, M.: The Veldkamp space of the smallest slim dense near hexagon. arXiv:0908.0989
12. Planat, M.: Three-qubit entangled embeddings of CPT and Dirac groups within E_8 Weyl group. arXiv:0906.1063
13. De Bruyn, B.: The hyperplanes of $DW(5, 2^h)$ which arise from embedding. *Discret. Math.* **309**, 304–321 (2009)
14. Pralle, H.: The hyperplanes of $DW(5, 2)$. *Exp. Math.* **14**, 373–384 (2005)
15. De Bruyn, B., Pralle, H.: The exceptional hyperplanes of $DH(5, 4)$. *Eur. J. Combin.* **28**, 1412–1418 (2007)
16. De Bruyn, B., Pralle, H.: The hyperplanes of $DH(5, q^2)$. *Forum Math.* **20**, 239–264 (2008)
17. Brouwer, A.E., Cuypers, H., Lambeck, E.W.: The hyperplanes of the M_{24} near polygon. *Graphs Combin.* **18**, 415–420 (2002)
18. De Bruyn, B., Shpectorov, S.: The hyperplanes of the $U_4(3)$ near hexagon. Preprint
19. De Bruyn, B., Pralle, H.: The hyperplanes of the near hexagon on the 2-factors of the complete graph K_8 . *Discret. Math.* **308**, 5656–5671 (2008)