PERGAMON

Chaos, Solitons and Fractals 13 (2002) 797–805

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Quadro-quartic Cremona transformations and four-dimensional pencil-space-times with the reverse signature

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Accepted 20 February 2001

Abstract

This paper is concerned with a couple of quadro-quartic Cremona transformations and their associated fundamental pencil-space-times. Both space-times are found to have four dimensions. Their signature is, however, *reverse* to what we observe, i.e. they feature three time dimensions and a single space coordinate only. The global structure of the three temporal dimensions is briefly examined. In a generic case, all the three are seen to be structurally equivalent. In a particular case, they exhibit an intriguing $(2+1)$ split-up. Strikingly, this is found to completely parallel the quadro-cubic case, except for the swapped roles of space and time. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

No phenomenon of natural science seems to be better grounded in our everyday experience than the fact that the world of macroscopic physical reality has three dimensions we call spatial and one dimension of a different character we call time. Although a tremendous amount of effort has been put so far towards achieving a plausible quantitative elucidation of and deep qualitative insight into the origin of these two puzzling numbers (see e.g. [1–10]), the subject still remains one of the toughest and most challenging problems faced by contemporary physics (and by other related fields of human inquiry as well). Perhaps the most thought-provoking approach in this respect is the one based on the concept of a transfinite, hierarchical fractal set usually referred to as the Cantorian space, $\mathscr{E}^{(\infty)}$ [11–30]. In its essence, $\mathscr{E}^{(\infty)}$ is an *in*finite-dimensional quasi-random geometrical space consisting of an *in*finite number of elementary (kernel) fractal sets; yet, the *expectation* values of its both topological and Hausdorff dimensions are *finite* and, in general, equal – amounting to 4.236067977. The latter fact motivated El Naschie [13,14] to think not only about the total dimensionality of space-time, but also about its enigmatic signature. His reasoning goes, loosely speaking, as follows. It is assumed that the effective topological dimension of $\mathscr{E}^{(\infty)}$, $\langle n\rangle$, grasps only spatial degrees of freedom, whereas its averaged Hausdorff dimension, $\langle d\rangle$, incorporates also the temporal part of the structure. These two dimensions are interconnected, as both depend on the Hausdorff dimension of the kernel set, $d_c^{(0)}$. And, when the resolution is at low energy, there exists a unique value of the latter, viz. $d_c^{(0)} = 1/2$, for which $\langle n\rangle = 3$ (space) and $\langle d\rangle = 4$ (space-time).

Partly motivated and guided by this nice probabilistic interpretation, we recently introduced a conceptually similar, but qualitatively different, algebraic geometrical method to successfully approach the issue [31,32]. Here, (the structure of) the macroscopic space-time is postulated to be identical with (that of) the so-called *fundamental* manifold of a Cremona transformation in a three-dimensional projective space. [1] Such a manifold, as demonstrated in detail in [31,32], is composed of pencils, i.e. of linear, singly-infinite aggregates of geometrical objects, each pencil

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[1] The paper employs the same definitions, symbols and notation as used in [31,32]: the reader is advised to consult first these two references in order to be able to fully comprehend the ideas discussed in what follows.

PII: S0960-0779(01)00054-6

being understood to represent one observable dimension of space-time. The difference between time and space finds here its natural expression in the fact that these 'fundamental' pencils are, in general, of *two* distinct kinds: the pencils consisting of lines, which are taken to generate spatial dimensions, and those made up of conics, which are adopted to stand for time coordinates. And there exist a remarkable couple of Cremona transformations, both of the so-called *quadro-cubic* type, whose associated fundamental manifolds exactly reproduce the total dimensionality of the universe (*four* fundamental pencils altogether) as well as its signature (*three* pencils of lines and only *one* pencil of conics).

As explicitly underlined in [31], the quadro-cubic Cremona transformations represent the simplest *a*symmetric birational correspondences between two three-dimensional projective spaces that are related, in one of the spaces, to a homaloidal web of quadrics. But there exists one more class of asymmetric space Cremona transformations linked with quadrics, viz. a *quadro-quartic* one. As already indicated by its name, the latter is generated by those homaloidal webs of quadrics whose inverse webs in the other projective space consist of *quartics*, i.e. surfaces of the *fourth* degree [33–35]. The paper aims at having a detailed look at the geometry of these transformations and inspecting basic structural properties of the corresponding fundamental pencil-space-times.

2. Quadro-quartic Cremona transformations and their fundamental manifolds

2.1. A generic case

The quadro-quartic Cremona transformations are induced, in one projective space (P_3), by homaloidal families of quadrics whose base system features four points, no three collinear, of which three ($B_i,\ i = 1,2,3$) are simple and the remaining one (B) is a point of contact [33–35]. We shall first consider a *generic* case, i.e. the case when all the four points are real and distinct. To facilitate our subsequent reasoning, we shall select the tetrahedron of reference in such a way [34] that the four base points are identical with its vertices, namely

$$B_1: \qquad \varrho\breve{z}_\alpha = (1,0,0,0), \tag{1}$$

$$B_2: \qquad \varrho\breve{z}_\alpha = (0,1,0,0), \tag{2}$$

$$B_3: \qquad \varrho\breve{z}_\alpha = (0,0,1,0), \tag{3}$$

$$B_4: \qquad \varrho\breve{z}_\alpha = (0,0,0,1), \tag{4}$$

where ϱ will henceforth always represent a non-zero proportionality factor, and the common tangent plane at B, Π^{T}, is given by

$$\Pi^{\mathrm{T}}: \qquad \breve{z}_1 + \breve{z}_2 + \breve{z}_3 = 0. \tag{5}$$

The equation of the corresponding web of quadrics then acquires a pretty simple form,

$$\mathscr{D}_{\vartheta}^{\diamondsuit}(\breve{z}) = \vartheta_1\breve{z}_2\breve{z}_3 + \vartheta_2\breve{z}_1\breve{z}_3 + \vartheta_3\breve{z}_1\breve{z}_2 + \vartheta_4\breve{z}_4(\breve{z}_1 + \breve{z}_2 + \breve{z}_3) = 0. \tag{6}$$

The web gives rise to the following Cremona transformation,

$$\varrho\breve{z}'_1 = \breve{z}_2\breve{z}_3, \tag{7}$$

$$\varrho\breve{z}'_2 = \breve{z}_1\breve{z}_3, \tag{8}$$

$$\varrho\breve{z}'_3 = \breve{z}_1\breve{z}_2, \tag{9}$$

$$\varrho\breve{z}'_4 = \breve{z}_4(\breve{z}_1 + \breve{z}_2 + \breve{z}_3), \tag{10}$$

sending the quadrics in question into the planes of the second projective space, P'_3. The inverse transformation, as easily verified, is given by

$$\varrho\breve{z}_1 = \breve{z}'_2\breve{z}'_3\Theta(\breve{z}'), \tag{11}$$

$$\varrho\breve{z}_2 = \breve{z}'_1\breve{z}'_3\Theta(\breve{z}'), \quad (12)$$

$$\varrho\breve{z}_3 = \breve{z}'_1\breve{z}'_2\Theta(\breve{z}'), \quad (13)$$

$$\varrho\breve{z}_4 = \breve{z}'_1\breve{z}'_2\breve{z}'_3\breve{z}'_4, \quad (14)$$

where

$$\Theta(\breve{z}') \equiv \breve{z}'_1\breve{z}'_2 + \breve{z}'_1\breve{z}'_3 + \breve{z}'_2\breve{z}'_3, \quad (15)$$

being related in P'_3 with a homaloidal family of (Steiner) *quartics*

$$\mathscr{P}^{\diamondsuit}_{\eta}(\breve{z}') = \eta_1\breve{z}'_2\breve{z}'_3\Theta(\breve{z}') + \eta_2\breve{z}'_1\breve{z}'_3\Theta(\breve{z}') + \eta_3\breve{z}'_1\breve{z}'_2\Theta(\breve{z}') + \eta_4\breve{z}'_1\breve{z}'_2\breve{z}'_3\breve{z}'_4 = 0. \quad (16)$$

These quartics, which are mapped by Eqs. (11)–(14) into the planes of P_3, all share three *lines* $\mathscr{L}^{B'}_k$, $k = 1, 2, 3$, namely

$$\mathscr{L}^{B'}_1 : \qquad \breve{z}'_2 = 0 = \breve{z}'_3, \quad (17)$$

$$\mathscr{L}^{B'}_2 : \qquad \breve{z}'_1 = 0 = \breve{z}'_3, \quad (18)$$

$$\mathscr{L}^{B'}_3 : \qquad \breve{z}'_1 = 0 = \breve{z}'_2, \quad (19)$$

each of them being singular and counting twice on $\mathscr{P}^{\diamondsuit}_{\eta}(\breve{z}')$, and a simple proper *conic* $\mathscr{Q}^{B'}$,

$$\mathscr{Q}^{B'} : \qquad \breve{z}'_4 = 0 = \breve{z}'_1\breve{z}'_2 + \breve{z}'_1\breve{z}'_3 + \breve{z}'_2\breve{z}'_3. \quad (20)$$

Again, our principal attention is focussed on the properties of the *fundamental* manifold associated with the web of quadrics, i.e. on the set of elements of P_3 that are sent by transformation (7)–(10) into the base points of $\mathscr{P}^{\diamondsuit}_{\eta}(\breve{z}')$ in P'_3. It is an easy exercise to find out that these elements are located in the planes $\breve{z}_k = 0$ $(k = 1, 2, 3)$, which correspond, respectively, to $\mathscr{L}^{B'}_k$ $(k = 1, 2, 3)$, as well as in the plane Π^{T}, which answers to $\mathscr{Q}^{B'}$. As in a generic quadro-cubic case [31], the fundamental manifold of P_3 comprises four distinct concurrent planes. Now, however, no three of them are linearly dependent. There exists another, more conspicuous, difference between the two cases as far as the nature of fundamental pencils in the four planes is concerned. For we find that now *three* planes feature pencils of *conics* and only *one* is endowed with a pencil of *lines* – exactly the converse of what characterizes the generic quadro-cubic case [31]. Let us see that explicitly [34].

Clearly, the pencil of fundamental elements in a given plane is the intersection of this plane with the individual quadrics of the web $\mathscr{D}^{\diamondsuit}_{\vartheta}(\breve{z})$. Thus, for the three planes $\breve{z}_k = 0$, we get

$$\tilde{\mathscr{Q}}^{(k)}_{\vartheta}(\breve{z}) : \qquad \vartheta_k\breve{z}_i\breve{z}_j + \vartheta_4\breve{z}_4(\breve{z}_i + \breve{z}_j) = 0, \quad (21)$$

where ijk is a permutation of 1, 2, 3, with $i \neq j \neq k$. The last equation represents, for a given value of k, a pencil of conics having three distinct real base points B_i, B_j and B, the latter being of multiplicity two, and a common tangent line $\mathscr{L}^{(k)}$, the meet of $\breve{z}_k = 0$ and Π^{T}. In order to see that a given conic of (21), $\tilde{\mathscr{Q}}^{(k)}_{\vartheta\equiv\vartheta_4/\vartheta_k}$, corresponds *as a whole* to a point of $\mathscr{L}^{B'}_k$, we first substitute $\breve{z}_k = 0$ into Eqs. (7)–(10), which gives

$$\breve{z}'_i = 0 = \breve{z}'_j \quad (22)$$

and

$$\varrho\breve{z}'_k = \breve{z}_i\breve{z}_j, \quad (23)$$

$$\varrho\breve{z}'_4 = \breve{z}_4(\breve{z}_i + \breve{z}_j), \quad (24)$$

and combine the last two equations with Eq. (21) to find

$$\breve{z}'_k = -\vartheta\breve{z}'_4, \quad (25)$$

which completes the proof and says that any conic of (21) is, indeed, a fundamental element in P_3. Turning next to the remaining plane, Π^{T}, and inserting its equation into Eq. (6), we obtain

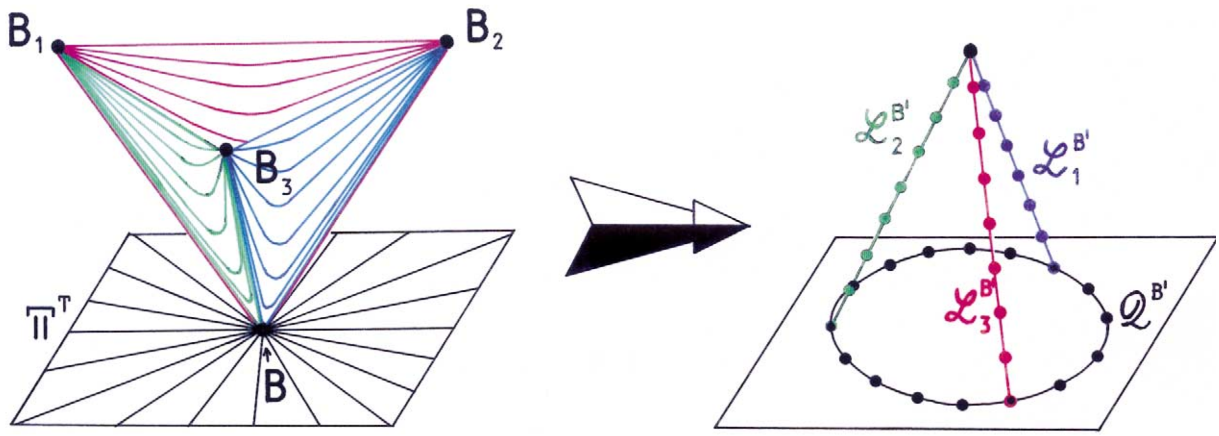

Fig. 1. Left: A sketch of the structure of the three pencils of fundamental conics and the pencil of fundamental lines associated with the homaloidal family of quadrics defined by Eq. (6); notice that the pencils of conics are all of the same type. Right: An outline of its 'primed' image, i.e. the configuration of the three base lines and the base conic of the inverse homaloidal web of quartics defined by Eq. (16); the lines are concurrent, non-coplanar, and each of them cuts the plane of the base conic at the point lying on the latter. The symbols and notation are explained in the text.

$$\tilde{\mathscr{L}}_{\vartheta}(\breve{z}): \qquad \vartheta_1\breve{z}_2\breve{z}_3 - \vartheta_2\breve{z}_3(\breve{z}_2+\breve{z}_3) - \vartheta_3\breve{z}_2(\breve{z}_2+\breve{z}_3) = -\vartheta_3\breve{z}_2^2 + (\vartheta_1-\vartheta_2-\vartheta_3)\breve{z}_2\breve{z}_3 - \vartheta_2\breve{z}_3^2 = 0, \tag{26}$$

which represents, as already announced, a pencil of lines having point B for their vertex. A generic line of this pencil, $\tilde{\mathscr{L}}(\kappa)$, can also be viewed as the intersection of Π^{T} with the plane

$$\breve{z}_2 = \kappa\breve{z}_3, \tag{27}$$

κ being a parameter. Putting the two equations together yields

$$\breve{z}_1 = -(\kappa+1)\breve{z}_3. \tag{28}$$

Next, a variable point $\breve{y}$ on $\tilde{\mathscr{L}}(\kappa)$ can, in the light of Eqs. (27), (28), be parametrized (in terms of μ) as

$$\varrho\breve{y}_\alpha(\mu) = (\kappa+1, -\kappa, -1, \mu) \tag{29}$$

and has its 'primed' counterpart, as implied by Eqs. (7)–(10), in the point

$$\varrho\breve{y}'_\alpha = (\kappa, -(\kappa+1), -\kappa(\kappa+1), 0). \tag{30}$$

The fact that the last expression is *in*dependent of the parameter μ means that nothing else than the *whole* line $\tilde{\mathscr{L}}(\kappa)$ is mapped into a single point of P'_3; and this point lies on the base conic $\mathscr{Q}^{B'}$, for

$$\breve{y}'_1\breve{y}'_2 + \breve{y}'_1\breve{y}'_3 + \breve{y}'_2\breve{y}'_3 = \frac{-\kappa(\kappa+1) - \kappa^2(\kappa+1) + \kappa(\kappa+1)^2}{\varrho^2} = \frac{-\kappa(\kappa+1)^2 + \kappa(\kappa+1)^2}{\varrho^2} \equiv 0. \tag{31}$$

Hence, any line in pencil (26) is a fundamental element of P_3, too. Both the fundamental manifold in P_3 and the base configuration in P'_3, together with the specific, one-way correspondence between them, are depicted in Fig. 1; as in [31,32], we have used different colours to make the portrayal more illustrative, reserving a particular colour to match the related geometric structures in the two projective spaces.

2.2. A particular case

The aggregate of quadrics (6) is, as already mentioned, the *most* general homaloidal web of quadrics in a three-dimensional real projective space whose inverse is of the fourth order, because its four base points are all distinct. There are, of course, a number of special cases which arise when two or more of these points coincide. Among them, of particular interest for us is that where one of the simple base points B_i, say B_3, lies infinitely close to the point of contact

B along a line, $\mathscr{L}^{\mathrm{T}}$, which is situated in Π^{T}. And it turns out [33] that this is the only other quadro-quartic case where the distinction between four fundamental planes is preserved.

With the view of making a subsequent comparison of the two quadro-quartic cases easy, we shall pick up the reference tetrahedron which has the base points B_1, B_2 and $B \equiv B_3$ at its corresponding vertices, that is [2]

$$B_1: \qquad \varrho\breve{z}_\alpha = (1,0,0,0), \tag{32}$$

$$B_2: \qquad \varrho\breve{z}_\alpha = (0,1,0,0), \tag{33}$$

$$B \equiv B_3: \qquad \varrho\breve{z}_\alpha = (0,0,1,0), \tag{34}$$

the (common tangent) line $\mathscr{L}^{\mathrm{T}}$ for one of its edges, namely

$$\mathscr{L}^{\mathrm{T}}: \qquad \breve{z}_1 = 0 = \breve{z}_2, \tag{35}$$

and the common tangent plane Π^{T} as one of its 'mirror' planes, viz.

$$\Pi^{\mathrm{T}}: \qquad \breve{z}_1 + \breve{z}_2 = 0. \tag{36}$$

Then,

$$\mathscr{D}_{\vartheta}^{\hat{\Diamond}}(\breve{z}) = \vartheta_1\breve{z}_2\breve{z}_4 + \vartheta_2\breve{z}_1\breve{z}_4 + \vartheta_3[\breve{z}_4^2 - (\breve{z}_1 + \breve{z}_2)\breve{z}_3] + \vartheta_4\breve{z}_1\breve{z}_2 = 0, \tag{37}$$

which furnishes us with the following $P_3 \to P_3'$ Cremona transformation:

$$\varrho\breve{z}_1' = \breve{z}_2\breve{z}_4, \tag{38}$$

$$\varrho\breve{z}_2' = \breve{z}_1\breve{z}_4, \tag{39}$$

$$\varrho\breve{z}_3' = \breve{z}_4^2 - (\breve{z}_1 + \breve{z}_2)\breve{z}_3, \tag{40}$$

$$\varrho\breve{z}_4' = \breve{z}_1\breve{z}_2. \tag{41}$$

Its $P_3' \to P_3$ dual is found to be

$$\varrho\breve{z}_1 = \breve{z}_2'\breve{z}_4'\Phi(\breve{z}'), \tag{42}$$

$$\varrho\breve{z}_2 = \breve{z}_1'\breve{z}_4'\Phi(\breve{z}'), \tag{43}$$

$$\varrho\breve{z}_3 = \breve{z}_1'\breve{z}_2'(\breve{z}_1'\breve{z}_2' - \breve{z}_3'\breve{z}_4'), \tag{44}$$

$$\varrho\breve{z}_4 = \breve{z}_1'\breve{z}_2'\Phi(\breve{z}'), \tag{45}$$

where

$$\Phi(\breve{z}') \equiv (\breve{z}_1' + \breve{z}_2')\breve{z}_4', \tag{46}$$

being coupled with

$$\mathscr{P}_{\eta}^{\hat{\Diamond}}(\breve{z}') = \eta_1\breve{z}_2'\breve{z}_4'\Phi(\breve{z}') + \eta_2\breve{z}_1'\breve{z}_4'\Phi(\breve{z}') + \eta_3\breve{z}_1'\breve{z}_2'(\breve{z}_1'\breve{z}_2' - \breve{z}_3'\breve{z}_4') + \eta_4\breve{z}_1'\breve{z}_2'\Phi(\breve{z}') = 0. \tag{47}$$

This homaloidal system of Steiner quartics also features, like its generic mate, three incident double lines,

$$\mathscr{L}_1^{B'}: \qquad \breve{z}_1' = 0 = \breve{z}_4', \tag{48}$$

$$\mathscr{L}_2^{B'}: \qquad \breve{z}_2' = 0 = \breve{z}_4', \tag{49}$$

[2] This system of homogeneous coordinates is slightly different from that used by Cremona [33].

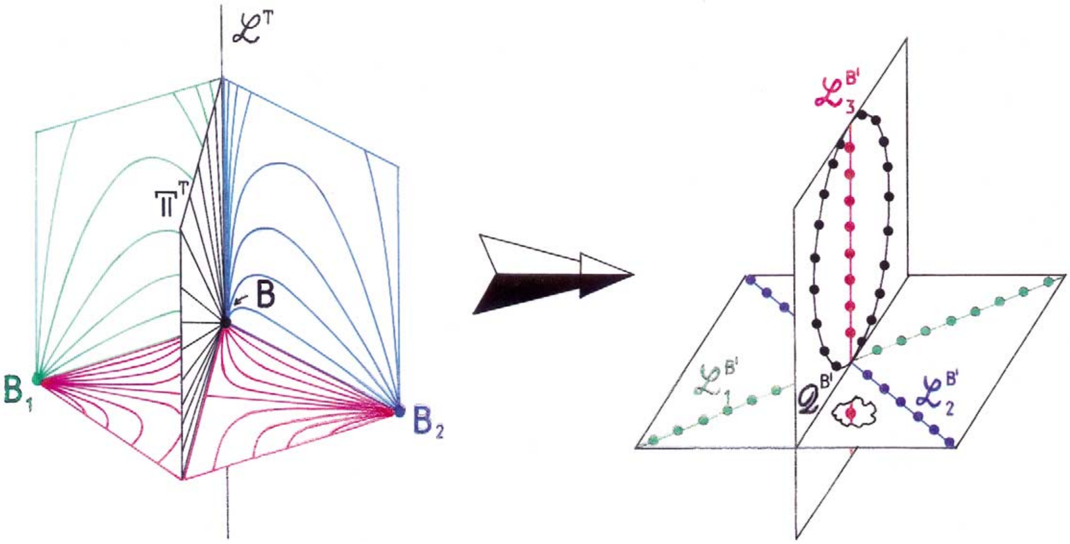

Fig. 2. Left: A sketch of the structure of the three pencils of fundamental conics and the pencil of fundamental lines associated with the homaloidal family of quadrics defined by Eq. (37); notice that the pencils of conics located in the planes $B_1\mathscr{L}^T$ and $B_2\mathscr{L}^T$ are of a different kind than that situated in the B_1B_2B plane. Right: An outline of its 'primed' image, i.e. the configuration of the three base lines and the base conic of the inverse homaloidal web of quartics defined by Eq. (47); note that the lines are concurrent at the point of the base conic, and one of them lies in the plane of the latter. Compare with Fig. 1. The symbols and notation are explained in the text.

$$\mathscr{L}_3^{B'}: \qquad \breve{z}'_1 = 0 = \breve{z}'_2, \tag{50}$$

and a single non-singular conic,

$$\mathscr{Q}^{B'}: \qquad \breve{z}'_1 + \breve{z}'_2 = 0 = \breve{z}'_1\breve{z}'_2 - \breve{z}'_3\breve{z}'_4, \tag{51}$$

as the base elements; yet here one of the base lines, namely $\mathscr{L}_3^{B'}$, is found to lie in the plane of the base conic, cutting the latter in two distinct points. This feature makes us suspect that the quadrics' related fundamental manifold will not look structurally as symmetric as the generic one. That this is, indeed, the case can easily be demonstrated.

Thus, the fundamental elements of P_3 are all located inside the planes $\breve{z}_1 + \breve{z}_2 = 0$ (i.e. Π^T), $\breve{z}_2 = 0$ ($B_1\mathscr{L}^T$), $\breve{z}_1 = 0$ ($B_2\mathscr{L}^T$), and $\breve{z}_4 = 0$ (B_1B_2B), for these are projected by transformation (38)–(41) into $\mathscr{Q}^{B'}$, $\mathscr{L}_1^{B'}$, $\mathscr{L}_2^{B'}$, and $\mathscr{L}_3^{B'}$, respectively – as displayed in Fig. 2. As in the preceding case, we find Π^T to be the site of a B-centred pencil of fundamental lines,

$$\tilde{\mathscr{L}}_\vartheta(\breve{z}): \qquad \vartheta_1\breve{z}_2\breve{z}_4 - \vartheta_2\breve{z}_2\breve{z}_4 + \vartheta_3\breve{z}_2^2 - \vartheta_4\breve{z}_4^2 = \vartheta_3\breve{z}_2^2 + (\vartheta_1 - \vartheta_2)\breve{z}_2\breve{z}_4 - \vartheta_4\breve{z}_4^2 = 0, \tag{52}$$

whilst each of the other three planes carries a pencil of fundamental conics,

$$\tilde{\mathscr{Q}}_\vartheta^{(1)}(\breve{z}): \qquad \vartheta_2\breve{z}_1\breve{z}_4 + \vartheta_3(\breve{z}_4^2 - \breve{z}_1\breve{z}_3) = 0, \tag{53}$$

$$\tilde{\mathscr{Q}}_\vartheta^{(2)}(\breve{z}): \qquad \vartheta_1\breve{z}_2\breve{z}_4 + \vartheta_3(\breve{z}_4^2 - \breve{z}_2\breve{z}_3) = 0, \tag{54}$$

and

$$\tilde{\mathscr{Q}}_\vartheta^{(3)}(\breve{z}): \qquad \vartheta_4\breve{z}_1\breve{z}_2 - \vartheta_3\breve{z}_3(\breve{z}_1 + \breve{z}_2) = 0, \tag{55}$$

respectively. These conics' pencils are, however, not all of the same type: in the first two cases, $\tilde{\mathscr{Q}}_\vartheta^{(1,2)}$, the conics *osculate* each other (i.e. have a three-point contact) at B, whereas in the third case, $\tilde{\mathscr{Q}}_\vartheta^{(3)}$, they only *touch* (i.e. have a simple, two-point contact) at B – as easily discernible from Fig. 2, left. This structural $(2+1)$ split-up among the pencils of fundamental *conics* of this particular quadro-quartic case immediately reminds us of a numerically identical geometrically-grounded factorization exhibited by the pencils of fundamental *lines* of the specific quadro-cubic case described in [32]. In fact, after comparing the findings of this section with the results of Section 3 of [32], we can even go further and claim that not only are the two cases seen to be the exact 'pencil' inverses of one another, in the very much the same manner as the two generic cases, but they are also found to be intimately interlinked with each other. This relationship can already be revealed and grasped by merely throwing a glance at the left-hand side of Fig. 2 and the same side of Fig. 1 of [32]. First, we observe that the two fundamental manifolds are globally identical, both comprising four distinct planes of which three are collinear (i.e. linearly dependent). Next, we focus our attention on Fig. 2, left, and imagine that from both B_1 and B_2 there is issued a tangent line to every fundamental conic of the corresponding pencils. Finally, we forget about these conics, being left with the configuration formally identical to that depicted in Fig. 1, left, of [32]! It is worth emphasizing that there is no analogue of this feature in the generic cases. Really, the generic quadro-quartic fundamental manifold consists, as shown in Section 2.1, of four distinct planes no three of which are linearly dependent (see Fig. 1, left), whereas in the corresponding quadro-cubic case there is a triple of fundamental planes, viz. those containing pencils of lines, that do share one and the same line (see Fig. 3, left, of [31]). This fact seems to positively contribute to our intuitive feeling, acquired in [32], that the particular, 'spatially factorized' quadro-cubic Cremonian space-time [32] might be a better candidate to mimic the observed macro-space-time than its generic, 'spatially homogeneous' relative [31].

3. The corresponding fundamental pencil-space-times

At this point, let us adopt the point of view of [31,32] and regard the couple of above-discussed fundamental manifolds as pencil-space-times, conceiving each pencil of fundamental lines/conics as a macroscopic space/time dimension. We then readily see that the corresponding space-times both have total dimensionality equal to four. Yet, featuring three time dimensions but only a single spatial one, their signature is exactly complementary to what is observed. Hence, the only reason why they deserve our attention is a technical one, in the following sense. It can easily be demonstrated that the generic quadro-cubic [31,32] and quadro-quartic Cremonian space-times are the *only* viable four-dimensional pencil-space-times associated with homaloidal webs of quadrics. And we have found that both kinds of them exhibit exclusively 'odd' signatures: either $(3+1)$ familiar to our senses (quadro-cubic cases [31,32]) or $(1+3)$ (unobserved; quadro-quartic cases). In other words, quadrics-related Cremona space-times are *in*compatible with 'even' signatures: $(4+0)$ (i.e. purely spatial manifolds), $(0+4)$ (purely temporal manifolds) and/or $(2+2)$. Moreover, since the quadro-*cubic* Cremona manifolds are geometrically of a much *simpler* character than their quadro-quartic siblings, the fact that the observed macro-space-time is endowed with one time and three spatial dimensions may reflect nothing else than Nature' fondness for symmetry and *simplicity* – a heuristic guiding principle playing so profound a role in contemporary physics [10,21].

The treatment of the subject would be incomplete if we skipped mentioning an intriguing property exhibited by the 'temporally anisotropic' quadro-quartic pencil-space-time. It goes without saying that its two time dimensions, generated by pencils (53) and (54), are of a different character than the third one, induced by pencil (55). What really fascinates here is, however, the fact that the planes carrying these pencils are *not* linearly dependent, as our experience with quadro-cubic space-times would seem to imply. [3] But this manifold, as already stressed, does possess three linearly dependent planes, viz. $B_1\mathscr{L}^{\mathrm{T}}$, $B_2\mathscr{L}^{\mathrm{T}}$, and Π^{T} (see Fig. 2, left). It is then obvious that one of them must contain the pencil of lines, as is indeed the case – the plane Π^{T}. What this means is that the time dimensions that are generated by the pencils of conics located in the planes $B_1\mathscr{L}^{\mathrm{T}}$ and $B_2\mathscr{L}^{\mathrm{T}}$ (i.e. by $\tilde{\mathscr{Q}}_{\vartheta}^{(1)}$ and $\tilde{\mathscr{Q}}_{\vartheta}^{(2)}$) are coupled to the (single) space dimension more closely than the remaining temporal dimension, generated by the conics' pencil lying in the 'odd' B_1B_2B plane (i.e. by $\tilde{\mathscr{Q}}_{\vartheta}^{(3)}$). Incidentally, it should be remarked that this 'standing apart' time dimension is of the same structure as the three temporal dimensions of the generic case, for pencil (55) is of the same projective type as those defined by Eq. (21); this fact is also easy to descry from Figs. 1 and 2, by comparing their left-hand sides.

Finally, there seems to also emerge an interesting link between our Cremonian view of space-time and transfinite (super)string theories. Just recently [36], we have noticed that the dimensional hierarchy within the so-called heterotic string space-times might be underlaid by a simple 'incidence' algebra of the configuration of 27 lines lying on a generic cubic surface in a three-dimensional projective space. If our assumption is correct, then the total dimensionality of bosonic string space-time is 27 instead of 26, and its supersymmetric sector has 12 dimensions, answering to the subgroup of lines known as Schläfli's double-six [36]. Here, it is the algebra of the lines that is our primary focus, for it is not an absolute property of the configuration as a whole, but rather a characteristic of the method of representation.

[3] In both the quadro-cubic cases [31,32], the three planes carrying the pencils of fundamental lines *are* linearly dependent.

And one of the most powerful and exceptionally illustrative representations employed is that based on a birational correspondence between the points of the cubic surface and the points of a projective plane; because this correspondence is of the *same* kind as any Cremona transformation, it naturally lends itself as a promising formal linking element between the two theories.

4. Concluding remarks

The paper has completed a basic qualitative treatment of the concept of Cremonian pencil-space-times introduced in [31] and further elaborated in [32]. Although the above-discussed quadro-quartic space-times do not correspond, in terms of their signature, to what the reality offers us, leaving them aside we would deprive ourselves of a better understanding of and an invaluable insight into the structure and conceptual setting of their quadro-cubic counterparts, which reproduce well classical space-time as we perceive it. Moreover, not having discussed these remarkable quadro-quartic configurations, we would not know, for example, that 'Cremonian' Nature favours exclusively odd numbers of both space and time dimensions, nor would we be aware that, theoretically, at least two globally distinct kinds of time dimension may coexist on one manifold.

Acknowledgements

I would like to thank Mr. P. Bendík for careful drawing of the figures. Some of this work was carried out during my extended stay at the University of California at Berkeley and the Noetic Advanced Studies Institute at Oakland (USA); I am very grateful to Dr. Richard L. Amoroso, the director of the latter institution, for his warm hospitality and financial support. The work was also supported in part by the NATO Collaborative Linkage Grant PST.CLG.976850.

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