



PERGAMON

Chaos, Solitons and Fractals 12 (2001) 2127–2142

CHAOS
SOLITONS & FRACTALS

www.elsevier.nl/locate/chaos

Cremona transformations and the conundrum of dimensionality and signature of macro-spacetime

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Accepted 2 August 2000

Abstract

The issue of dimensionality and signature of the observed universe is analysed. Neither of the two properties follows from first principles of physics, save for a remarkably fruitful Cantorian fractal spacetime approach pursued by El Naschie, Nottale and Ord. In the present paper, the author's theory of pencil-generated spacetime(s) is invoked to provide a clue. This theory identifies spatial coordinates with pencils of lines and the time dimension with a specific pencil of conics. Already its primitive form, where all pencils lie in one and the same projective plane, implies an intricate connection between the observed multiplicity of spatial coordinates and the (very) existence of the arrow of time. A qualitatively new insight into the matter is acquired, if these pencils are not constrained to be coplanar and are identified with the pencils of *fundamental* elements of a Cremona transformation in a projective space. The correct dimensionality of space (3) and time (1) is found to be uniquely tied to the so-called *quadro-cubic* Cremona transformations – the *simplest* non-trivial, non-symmetrical Cremona transformations in a projective space of three dimensions. Moreover, these transformations also uniquely specify the type of a pencil of fundamental conics, i.e. the global structure of the time dimension. Some physical and psychological implications of these findings are mentioned, and a relationship with the Cantorian model is briefly discussed. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

Our universe is endowed with a large number of remarkable features of which its dimensionality can, undoubtedly, be ranked among those of the top importance. It has been confirmed in innumerable ways that, at a *macroscopic* level, the physical world has four dimensions. Not all the four are, however, equivalent: three of them, we call spatial, are found to stand on a different footing than the remaining one, the time dimension. Time and space are inseparable, forming a continuum referred to as spacetime. Yet, they do considerably differ from each other, the difference being much more pronounced at the perceptual than physical level. Physics, in its current state-of-the-art form, is not only clueless as for the total multiplicity of the dimensions, but it also lacks any reasonable explanation why there should be two distinct kinds of them, and combined in a puzzling 3:1 ratio at that. In other words, *neither* the dimensionalists *nor* the signature of the observed universe follows from first principles of physics: these are free parameters to be fixed observationally.

Yet, there have been numerous speculations to explain these two features using heuristic methods. Among physically based arguments, it is worth mentioning the well-known Weyl's observation that the Maxwell equations are tied uniquely to the 3 + 1 spacetime, and/or intriguing Ehrenfest's reasoning that stable atoms are only possible in 3 + 1 dimensions [1]. Another class of heuristic inquiries is more

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mathematically-oriented. Here we can, for example, rank the fact that the Weyl tensor, which in Einstein's gravitation theory carries information about that part of the spacetime curvature which is not locally determined by the energy–momentum, vanishes in less than four dimensions [2], or a topological reason that $n \neq 4$ dimensional manifolds always feature a unique differentiable structure, while those with $n = 4$ do not [3]. Finally, there is a large and still growing group of scholars who favour the so-called anthropic principle for a rational explanation of the macro-dimensionality of spacetime [4].

A principally new and mathematically rigorous method to tackle this question was adopted by Ord [5], Nottale [6] and most actively pursued by El Naschie [7–13]. According to the last-mentioned author, the apparent dimensionality of our macro-spacetime is simply equal to a *finite* expectation value of an otherwise *infinite*-dimensional transfinite fractal set called the Cantorian space, $\mathcal{E}^{(\infty)}$ [7–13]. What strikes most about this approach is the fact that it also provides a couple of intriguing clues about the possible origin of the observed *signature* of spacetime [7,9,10,12]. The first argument is based on the finding that, under a special assumption, the effective topological dimension of $\mathcal{E}^{(\infty)}$, $\langle n \rangle = 3$, differs from its averaged Hausdorff dimension, $\langle d \rangle = 4$. The author then argues that the distinction between space and time may simply reflect the fact that our direct physical access is limited to $\langle n \rangle = 3$ dimensions (space), so that the remaining dimension (time), $\langle d \rangle - \langle n \rangle = 1$, can be only felt, perceived mentally [7]. The other case is connected with a space of hyperspheres, $\mathcal{S}^{(\infty)}$ [9,10]. This space can be viewed as an infinite collection of unit hyperspheres with any conceivable dimension. As the volume of an n -dimensional unit sphere vanishes with n tending to infinity, $\mathcal{S}^{(\infty)}$ has, like $\mathcal{E}^{(\infty)}$, a finite effective dimension which is equal to 4. Yet, its 'volume' appearance is that of a classical *three*-dimensional sphere. This difference is seen as another justification of the observed factorization of 4D into $(3 + 1)$ D [9,10].

Apart from this intriguing and fruitful number-probabilistic approach, there also exists what can be termed an algebro-geometrical elucidation of the origin of the observed macro-dimensionality/macro-signature of spacetime. The latter is based on our theory of pencil-generated spacetimes [14–23]. This 'pencil' theory was originally motivated by and aimed at a deeper insight into a puzzling discrepancy between perceptual and physical aspects of time. Yet, we soon realized that it has also something to say to the problem in question. Namely, we found out an intricate connection between the observed multiplicity of spatial dimensions and the appearance of a non-trivial internal structure ('arrow') of time [14–16,20,21]. Mathematically, this is substantiated by the fact that we treat time and space as standing on topologically different footings. As for their 'outer' appearance, both the types of dimensions are identical, being regarded as *pencils*, i.e. linear single-parametrical aggregates, of constituting elements. It is their 'inner' structure where the difference comes in: thus, the constituting element ('point') of a spatial dimension is a *line*, whereas that of the time dimension is a proper *conic* [14–16,20,21]. It may come as a surprise to the reader to find out that the theory which offers us such important hints on the macro-dimensionality/macro-signature of spacetime requires nothing more than a projective plane, i.e. a projective space of *two* dimensions for its operational framework. Hence, one may wonder what happens if one lifts the dimensionality of this projective setting by 1, i.e. if one moves into a projective space of *three* dimensions, allowing so different pencils to lie in different planes. It is the subject of the present paper to show that relaxing the constraint of 'coplanarity' gives indeed our theory a sufficiently extended framework to substantially broaden and deepen our understanding of the dimensionality and signature of the observed universe.

2. Pencil-spacetime(s) having as a background the projective plane

2.1. Projective plane and its basic properties

We shall start by giving an overview of the pencil concept of spacetime; although we shall try to make this account as self-contained as possible, the reader wishing to go into more details is referred to consult our papers [14–19,23].

As already mentioned, the corner-stone of the concept is a projective plane, P_2 . There are a number of ways to define this remarkable manifold [24,25]. The oldest and perhaps conforming best to our intuition is its view as the familiar Euclidean plane extended/augmented by a single line, called the 'ideal' line, or the line 'at infinity.' The addition of the ideal line *closes* the projective plane, making it *non-orientable* (i.e.

one-sided like a famous Möbius band) and endowing it with many other features not exhibited by the original Euclidean plane. Equivalently, the projective plane can be regarded as a set of all lines in the usual three-dimensional Euclidean space, E_3 , which concur in (pass through) a point, i.e. as a star of lines. Each line in the star, representing a point of P_2 , is uniquely specified by its intersection with a unit sphere enclosing the point of concurrence. Since this line pierces the sphere twice, the two points must be treated as identical. So, we have another model of P_2 as a unit sphere in E_3 whose antipodal points have been identified. Topologically speaking, the projective plane can be constructed by gluing together both pairs of opposite edges of a rectangle, giving them a half-twist. Due to its closeness and non-orientability, the projective plane cannot be embedded in E_3 , i.e. presented there as a smooth surface without self-intersections. Yet, it can be represented as a surface having self-intersections and/or singular points. A remarkable class of such surfaces are the so-called Steiner surfaces [25,26]. Another interesting image of the projective plane in E_3 is the so-called Boy surface [25].

In order to increase familiarity with the concept and reveal further details of the structure of the projective plane, it is necessary to perform its coordinatization. To this end, we return to the star representation of P_2 discussed above. Selecting in E_3 the Cartesian coordinate system x, y, z in such a way that its origin coincides with the vertex of the star, a line of the star is simply given by

$$ax + by + cz = 0, \tag{1}$$

where a, b and c are constants of which at least one is non-zero. Clearly, the triple (a, b, c) specifies the line uniquely, as does any proportional triple (qa, qb, qc) , $q \neq 0$. Recalling that the lines of the star are in a one-to-one correspondence with the points of P_2 , we see that the latter may also be represented in the same way. Thus, the projective plane contains as points all triples $(\check{x}_1, \check{x}_2, \check{x}_3)$,¹ disallowing the $(0, 0, 0)$ one, with proportional triples representing the same point; because scaling is unimportant, the coordinates \check{x}_i , $i = 1, 2, 3$, are called *homogeneous* coordinates.

Now, a *line* $\mathcal{L}(\check{x})$ in the projective plane is defined as the locus (i.e. the totality) of points \check{x}_i satisfying a *linear* equation of the form

$$\mathcal{L}(\check{x}) \equiv \sum_{i=1}^3 \zeta_i \check{x}_i \equiv \zeta_1 \check{x}_1 + \zeta_2 \check{x}_2 + \zeta_3 \check{x}_3 = 0, \tag{2}$$

where the coefficients ζ_i are not all zero. Since two linear equations have the same locus if and only if their coefficients are proportional, the ordered triple $(\zeta_1, \zeta_2, \zeta_3)$ can be taken as *homogeneous* coordinates of the line $\mathcal{L}(\check{x})$. This is not the result of coincidence for, in P_2 , there is a perfect symmetry between points and lines, the two sets being regarded as *dual* to each other. This duality concept is another important property of the projective plane that finds no analogue in the ordinary Euclidean geometry. A *quadratic* equation

$$\begin{aligned} \mathcal{Q}(\check{x}) &\equiv \sum_{i,j=1}^3 c_{ij} \check{x}_i \check{x}_j \\ &\equiv c_{11} \check{x}_1^2 + c_{22} \check{x}_2^2 + c_{33} \check{x}_3^2 + 2c_{12} \check{x}_1 \check{x}_2 + 2c_{13} \check{x}_1 \check{x}_3 + 2c_{23} \check{x}_2 \check{x}_3 = 0, \end{aligned} \tag{3}$$

where the coefficients $c_{ij} = c_{ji}$ are not all zero, defines a *conic*. The conic may be irreducible (proper), or reducible (composite) accordingly as $\det(c_{ij})$ deviates from zero or not. As for the former, we distinguish between real and imaginary (i.e. having an empty image), while the latter comprise a pair of (real or conjugate complex, distinct or coincident) lines (see, e.g., [27]).

Both lines and conics can be found to form aggregates of different orders and with a various degree of complexity. The simplest of them, viz. linear and singly infinite, are usually referred to as *pencils*. Thus, a couple of distinct lines $\mathcal{L}^{(1)}(\check{x})$ and $\mathcal{L}^{(2)}(\check{x})$ define a pencil

¹ We change to this accented subscript notation in order to be compatible with the notation adopted in our previous papers [14–23]. Also, if not stated otherwise, the ground field of the projective plane/space will be assumed to be that of the real numbers.

$$\mathcal{L}_\vartheta(\check{x}) \equiv \vartheta_1 \mathcal{L}^{(1)}(\check{x}) + \vartheta_2 \mathcal{L}^{(2)}(\check{x}) = \vartheta_1 \sum_{i=1}^3 \zeta_i^{(1)} \check{x}_i + \vartheta_2 \sum_{i=1}^3 \zeta_i^{(2)} \check{x}_i = 0, \tag{4}$$

with $(\vartheta_1, \vartheta_2)$ being any real couple except for $(0, 0)$, which obviously consists of all the lines passing through the point shared by the two lines; the point in question is called a *vertex* of the pencil. Similarly, given two different (not necessarily proper) conics $\mathcal{Q}^{(1)}(\check{x})$ and $\mathcal{Q}^{(2)}(\check{x})$, we have a pencil

$$\mathcal{Q}_\vartheta(\check{x}) \equiv \vartheta_1 \mathcal{Q}^{(1)}(\check{x}) + \vartheta_2 \mathcal{Q}^{(2)}(\check{x}) = \vartheta_1 \sum_{i,j=1}^3 c_{ij}^{(1)} \check{x}_i \check{x}_j + \vartheta_2 \sum_{i,j=1}^3 c_{ij}^{(2)} \check{x}_i \check{x}_j = 0. \tag{5}$$

The situation here is, however, much more complex than in the case of lines. For two distinct conics have four points in common, some or all of which may be imaginary and/or coincident; hence, we find as many as nine different kinds of pencils of conics in the real projective plane [16,27]. The four points of intersection of two given conics which determine the pencil clearly belong to every conics in the pencil, and they are usually called the *base points*. Evidently, there is just one conic of the pencil through any point $\check{x}_i = \check{x}_i^\circ$ of the plane other than a base point. For, as \check{x}_i° is not a base point, at least one of the numbers $\mathcal{Q}^{(1)}(\check{x}^\circ)$, $\mathcal{Q}^{(2)}(\check{x}^\circ)$ is non-zero. Therefore, the equation $\mathcal{Q}^{(1)}(\check{x}^\circ) + \vartheta \mathcal{Q}^{(2)}(\check{x}^\circ) = 0$ determines the parameter $\vartheta \equiv \vartheta_2/\vartheta_1$, and so the conic, uniquely.

2.2. Pencil concepts of space and time

As the elements of any pencil are fully described in terms of a single parameter (ϑ), this kind of aggregate can be regarded, on par with a line, as the *simplest* one-dimensional geometrical structure of the projective plane. It should not, therefore, come as a surprise that we postulate both the spatial and temporal dimensions to be represented by pencils [14–16]. That is, at the very abstract level there is no distinction between time and space in our approach. The difference between the two concepts appears as soon as the character of the elements of the corresponding pencils is concerned; for a spatial coordinate is taken to be generated by a pencil of *lines*, while the temporal dimension is induced by a pencil of *conics*. In other words, a ‘point’ of space is represented by a projective line, whereas a ‘point’ of time (an ‘event’) is represented by a proper projective conic. Space, in our pencil view, is thus a simpler concept than time.

In general, any pencil of lines can serve as a spatial dimension and, similarly, any pencil of conics can stand for the temporal coordinate. This original symmetry is broken after one fixes a particular ‘temporal’ pencil of conics and postulates that only those pencils of lines can generate spatial coordinates, whose vertices lie on the composite conics of the conics’ pencil selected (such line-pencils will be referred to as *s-pencils*). As there is only one kind of proper projective conic with a non-empty image, our temporal dimension is still internally structureless. The final task, that is endowing time with its arrow-like structure, is simply accomplished by ‘dehomogenizing’ the projective plane. This can be furnished, in particular, by picking up one line (henceforth referred to as the *d-line*) and giving it a distinguished footing among the other lines; stipulating, at the same time, that out of all potential spatial dimensions we can observe only those represented by the *s-pencils* whose vertices fall on this line. It is namely here where the already-mentioned intriguing connection between the appearance of time’s arrow and the number of the observed spatial coordinates emerges [14–16,20,21].

2.3. Arrow of time and dimensionality of space

In order to see this feature explicitly, we shall consider the following pencil of conics:

$$\mathcal{Q}_\vartheta^{\check{x}}(\check{x}) = \vartheta_1 \check{x}_1 \check{x}_2 + \vartheta_2 \check{x}_3^2 = 0 \tag{6}$$

which, for many reasons [14,16–21], is our favourable time representative. This pencil features (see Fig. 1) two distinct base points $B_1 : \varrho \check{x}_i = (0, 1, 0)$ and $B_2 : \varrho \check{x}_i = (1, 0, 0)$, each of multiplicity 2, and a couple of composite conics, viz. a double real line $\check{x}_3 = 0$ (i.e., $B_1 B_2$) for $\vartheta = \pm\infty$, and a pair of real lines $\check{x}_1 = 0$ ($B_1 S$) and $\check{x}_2 = 0$ ($B_2 S$) that corresponds to $\vartheta = 0$; the point $S : \varrho \check{x}_i = (0, 0, 1)$ being the meet of the two lines.

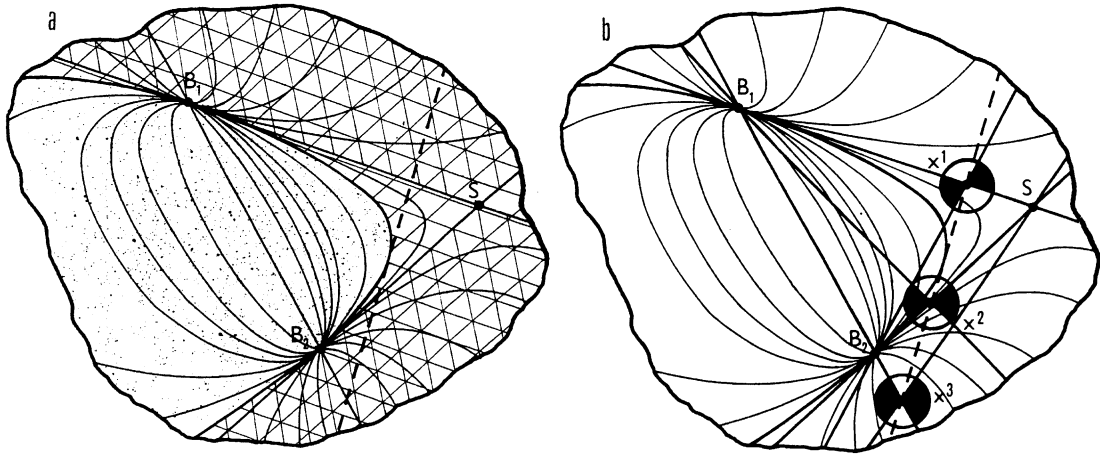


Fig. 1. The structure of the temporal arrow (a) and the corresponding multiplicity of spatial coordinates (b) of a spacetime configuration generated by a ‘regularly’ dehomogenized pencil of conics defined by Eq. (6). The symbols and notation are introduced in the text.

These degenerates separate the set of proper conics into two distinct families: $-\infty < \vartheta < 0$ and $0 < \vartheta < +\infty$.

Let us now dehomogenize the pencil by choosing the d -line (dashed in Fig. 1) in such a way that it incorporates neither of the base points $B_{1,2}$, nor the point S . It is obvious that the most general equation of a line meeting such constraints reads

$$\check{x}_1 - m\check{x}_2 - n\check{x}_3 = 0 \tag{7}$$

if both m and n are *non-zero* (assumed, without any substantial loss of generality, to be positive). Inserting this equation into Eq. (6) yields

$$P(x) \equiv mx^2 + nx + \vartheta = 0, \tag{8}$$

where we put $x \equiv \check{x}_2/\check{x}_3$; this *quadratic* equation has the roots

$$x_{\pm} = \frac{-n \pm \sqrt{n^2 - 4\vartheta m}}{2m}. \tag{9}$$

Now, as both m and n are fixed quantities, the value and character of the roots, that is, the intersection properties of a conic of the pencil and the d -line, depend solely on the value of the parameter ϑ . The proper conics of (6) are thus seen to form, with respect to d -line, two distinct domains: the domain consisting of conics having with this line two different real points in common (x_{\pm} distinct and real; the ‘cutters’ – the conics located in the shaded area in Fig. 1(a))

$$0 < \vartheta < n^2/4m \tag{10}$$

as well as the domain featuring conics having with it no real intersections (x_{\pm} distinct but imaginary; the ‘non-cutters’ – the curves occupying the dotted area in Fig. 1(a))

$$n^2/4m < \vartheta < +\infty; \tag{11}$$

the two regions being separated from each other by the *single* proper conic having d -line for a tangent ($x_+ \equiv x_-$; the ‘toucher’ – the curve shown bold in Fig. 1(a))

$$\vartheta = n^2/4m. \tag{12}$$

And this is really a very remarkable pattern because it reproduces strikingly well, at least at a qualitative level, the observed arrow of time after we postulate [14–16] that *the events of the past/future are represented by the cutters/non-cutters, and that the toucher stands for the unique moment of the present.*

In addition to being a means that enables us, as we have just shown, to endow the temporal coordinate with a non-trivial internal structure, the above-described dehomogenization of the projective plane also induces a very interesting sort of non-equivalence among the potential spatial dimensions, i.e. among the *s*-pencils. Clearly, in the case under discussion, any pencil of lines whose vertex falls on one of the lines B_1S , B_2S or B_1B_2 is an *s*-pencil. So, there is an infinite number of them. Yet, only *three* of them, viz.

$$\mathcal{L}_{\varpi_1}(\check{x}) = \check{x}_1 + \varpi_1(\check{x}_1 - m\check{x}_2 - n\check{x}_3) = 0, \tag{13}$$

$$\mathcal{L}_{\varpi_2}(\check{x}) = \check{x}_2 + \varpi_2(\check{x}_1 - m\check{x}_2 - n\check{x}_3) = 0, \tag{14}$$

and

$$\mathcal{L}_{\varpi_3}(\check{x}) = \check{x}_3 + \varpi_3(\check{x}_1 - m\check{x}_2 - n\check{x}_3) = 0, \tag{15}$$

where the parameters ϖ_i , $i = 1, 2, 3$, run through all the real numbers and infinity, possess the property of incorporating *d*-line. If we further suppose that only such line-pencils can generate the *observable* spatial dimensions, we arrive at a nice elucidation of the correct macro-dimensionality of space. The situation is illustrated in Fig. 1(b), where these pencils are represented by the ‘half-filled-up’ circles; we see, in particular, that two spatial coordinates (denoted by x^1 and x^2) are borne by the $\vartheta = 0$ degenerate, while the remaining one (x^3) is supported by the other, the $\vartheta = \pm\infty$ composite. As it can also be easily discerned from Fig. 1, this unique relation ‘arrow of time \leftrightarrow three-dimensionality of space’ holds only in the case if *d*-line is not incident with any of the points S , B_1 and B_2 . With this restriction dropped, the arrow of time disappears and space either loses one dimension or acquires infinitely many [16]; we enter the realm of singular/degenerate spacetimes which, although quite bizarre and unusual, represent the observable aspects of Nature, too! This fascinating topic lies, however, far outside the scope of the present paper and its in-depth discussion can be found elsewhere [20,21,23].

Although the correct macro-dimensionality of space is found to follow rather naturally from our pencil-formalism, it is not so with time: as the attentive reader may have noticed, the fact that there is just a single temporal dimension is still a matter of postulation. And this is a serious shortcoming of the theory indeed. Yet, there exists an easy way around. A crucial step to be made is to simply increase the dimensionality of our projective setting by 1, i.e. to go into a projective *space*. This move will not only enable us to generalize the pencil concept of spacetime in a fashion that obviates the problem mentioned, but it will also shift us somewhat closer to the spirit of how a physicist tackles the problem. In particular, we shall find that a special kind of birational transformations in the projective space, called Cremona transformations, may even provide us with a much sought-after elementary connection between the physical description of spacetime and the way we perceive it.

3. Pencil-spacetime(s) residing in the projective space

3.1. Rudiments of the projective space

As almost all of the concepts we have introduced in the projective plane, P_2 , can straightforwardly be extended to the projective space, P_3 , we shall pass over this part of the theory in a rather informative way. Thus, with $(0, 0, 0, 0)$ prohibited, we define P_3 as the set of all quadruples $(\check{z}_1, \check{z}_2, \check{z}_3, \check{z}_4)$, where the proportional quadruples are identified. Slightly rephrased, \check{z}_α , $\alpha = 1, 2, 3, 4$, can be regarded as the homogeneous coordinates of a *point* of P_3 . The most naive way of picturing the projective space is that of the usual Euclidean space, E_3 , augmented by a special plane, called the ideal plane, or the plane at infinity (see, e.g., [28,29]). A plane, $\Pi(\check{z})$, is defined as the locus of the points \check{z}_α satisfying a linear equation

$$\Pi(\check{z}) \equiv \sum_{\alpha=1}^4 \Pi_\alpha \check{z}_\alpha \equiv \Pi_1 \check{z}_1 + \Pi_2 \check{z}_2 + \Pi_3 \check{z}_3 + \Pi_4 \check{z}_4 = 0. \tag{16}$$

This equation is determined by its coefficients Π_α , which may clearly be taken as a quadruple of *homogeneous* coordinates of the plane. So, in a projective space, a point and a plane are *dual* concepts. A line is

seen to be a *self-dual* concept, since it is both the join of two points or the intersection of two planes. A point is incident with (or belongs to) a plane if its coordinates satisfy the equation of the plane, and it is incident with a line when its coordinates simultaneously satisfy two equations defining the line. From these two fundamental incidence relations, we get readily numerous incidence properties of planes and lines. Thus a line either has a unique point in common with a plane, or it lies completely in it. Two distinct planes share a unique line. Three distinct planes have either a single point, or a line in common. Two lines either lie in a common plane, in which case they also have a unique common point, or they neither lie in a common plane, nor possess a common point; in the latter case they are said to be *skew*.

Not only the points and the planes of P_3 , but also the lines can be represented by suitably chosen sets of homogeneous coordinates. Let us consider a generic line p of P_3 . The line is uniquely defined by any two of its points. Taking these as $\check{z}_\alpha^{(1)}$ and $\check{z}_\alpha^{(2)}$, we define the quantities

$$p_{\alpha\beta} = \check{z}_\alpha^{(1)}\check{z}_\beta^{(2)} - \check{z}_\beta^{(1)}\check{z}_\alpha^{(2)}, \tag{17}$$

which are called the *Plücker homogeneous coordinates* of the line p ; the fundamental property of these line-coordinates, as it is fairly easy to see, is that their ratios are independent of the choice of the two points on p . There are altogether 16 quantities $p_{\alpha\beta}$, but since they are skew-symmetric (i.e., $p_{\alpha\beta} = -p_{\beta\alpha}$) their effective number is reduced to 6. These six quantities are, however, not independent, for there are ∞^5 sets of ratios of them, but only ∞^4 lines in P_3 . The Plücker coordinates of every line must therefore be connected by one and the same identical relation, viz.

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \tag{18}$$

Instead of determining a line by two of its points, it may be determined by two of its planes, say $\Pi_\alpha^{(1)}$ and $\Pi_\alpha^{(2)}$. Putting

$$\pi_{\alpha\beta} = \Pi_\alpha^{(1)}\Pi_\beta^{(2)} - \Pi_\beta^{(1)}\Pi_\alpha^{(2)}, \tag{19}$$

then by duality of the previous argument, a set of quantities $\pi_{\alpha\beta}$, not all zero and satisfying

$$\pi_{12}\pi_{34} + \pi_{13}\pi_{42} + \pi_{14}\pi_{23} = 0 \tag{20}$$

also determine the line uniquely. There is thus a second, and equally justified, set of homogeneous coordinates of the line. The two systems are, however, not independent, for obviously ²

$$\frac{p_{12}}{\pi_{34}} = \frac{p_{13}}{\pi_{42}} = \frac{p_{14}}{\pi_{23}} = \frac{p_{34}}{\pi_{12}} = \frac{p_{42}}{\pi_{13}} = \frac{p_{23}}{\pi_{14}}. \tag{21}$$

Finally, let us find the condition when two lines p and q intersect. If the ray coordinates of the lines are given by their respective point pairs $\check{z}_\alpha^{(1)}, \check{z}_\alpha^{(2)}$ and $\check{z}'_\alpha, \check{z}''_\alpha$, then the lines are incident iff the four points are coplanar, which amounts to

$$\det \begin{pmatrix} \check{z}_1^{(1)} & \check{z}_2^{(1)} & \check{z}_3^{(1)} & \check{z}_4^{(1)} \\ \check{z}_1^{(2)} & \check{z}_2^{(2)} & \check{z}_3^{(2)} & \check{z}_4^{(2)} \\ \check{z}'_1 & \check{z}'_2 & \check{z}'_3 & \check{z}'_4 \\ \check{z}''_1 & \check{z}''_2 & \check{z}''_3 & \check{z}''_4 \end{pmatrix} = 0. \tag{22}$$

Expanding this determinant and making use of Eq. (17), the condition of incidence reads

$$\Theta(p, q) \equiv p_{12}q_{34} + p_{13}q_{42} + p_{14}q_{23} + p_{34}q_{12} + p_{42}q_{13} + p_{23}q_{14} = 0, \tag{23}$$

² In order to distinguish between the two kinds of Plücker coordinates, the former (Eq. (17)) are usually called *ray* line-coordinates, while the latter (Eq. (19)) are, as a rule, termed *axial*.

which, in a dualized form, looks like

$$\tilde{\Theta}(\pi, \kappa) \equiv \pi_{12}\kappa_{34} + \pi_{13}\kappa_{42} + \pi_{14}\kappa_{23} + \pi_{34}\kappa_{12} + \pi_{42}\kappa_{13} + \pi_{23}\kappa_{14} = 0. \tag{24}$$

Incidentally, we observe that in a special case of $p \equiv q$ Eqs. (23) and (24) reduce, respectively, into Eqs. (18) and (20), as expected.

Next, a surface of P_3 is the totality of points which have two degrees of freedom, and it may be defined by imposing a single analytical constraint on a generic point. If this constraint takes the form of a homogeneous *quadratic* equation

$$\begin{aligned} \mathcal{D}(\check{z}) &\equiv \sum_{\alpha, \beta=1}^4 d_{\alpha\beta} \check{z}_\alpha \check{z}_\beta \\ &\equiv + d_{11} \check{z}_1^2 + d_{22} \check{z}_2^2 + d_{33} \check{z}_3^2 + d_{44} \check{z}_4^2 + 2d_{12} \check{z}_1 \check{z}_2 + 2d_{13} \check{z}_1 \check{z}_3 \\ &\quad + 2d_{14} \check{z}_1 \check{z}_4 + 2d_{23} \check{z}_2 \check{z}_3 + 2d_{24} \check{z}_2 \check{z}_4 + 2d_{34} \check{z}_3 \check{z}_4 = 0, \end{aligned} \tag{25}$$

the surface is called a *quadric* surface, or a *quadric* for short; it is clearly analogous to a conic of P_2 (see Eq. (3)). A quadric is *proper* or *degenerate* according as $\det(d_{\alpha\beta})$ deviates from zero or not; in the latter case, there always exists a coordinate system in which the form $\mathcal{D}(\check{z})$ features fewer than four variables. As for real proper quadrics, here we distinguish between ruled and non-ruled according as they do or do not contain lines, respectively. The equation of a ruled quadric can always be reduced into a simple canonical form

$$\widehat{\mathcal{D}}(\check{z}) = \check{z}_1 \check{z}_4 - \check{z}_2 \check{z}_3 = 0, \tag{26}$$

from where it readily follows that such a quadric contains two distinct, singly infinite systems of lines (called generators), viz. the λ -system

$$\check{z}_1 - \lambda \check{z}_3 = 0 = \check{z}_2 - \lambda \check{z}_4 \tag{27}$$

and the μ -system

$$\check{z}_1 - \mu \check{z}_2 = 0 = \check{z}_3 - \mu \check{z}_4, \tag{28}$$

as illustrated in Fig. 2; here, both λ and μ are variable parameters that can acquire any real value and infinity as well. We leave it to the reader to verify that a line from one system is incident with any line of the other system, and that any two distinct lines belonging to the same system are skew (both the features being easily discernible from Fig. 2).

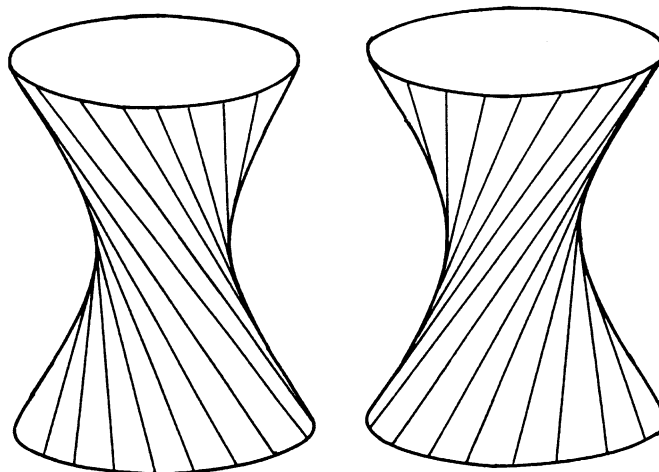


Fig. 2. The two complementary systems (reguli) of line generators lying on a ruled quadric surface (represented here as a hyperboloid of one sheet).

3.2. Homaloidal webs and Cremona transformations

In this subsection we shall introduce a very important new concept, that of the so-called Cremona transformations, which will enable us to bring together under a common head the planar pencil-concepts of time and space. In addition, it will serendipitously lend itself as an intriguing means of unifying the physical aspects of spacetime with their perceptual/psychological counterparts.

To this end, we shall consider a *web*, i.e. a linear, triply-infinite aggregate of quadrics $\mathcal{D}_\vartheta(\check{z})$:

$$\mathcal{D}_\vartheta(\check{z}) \equiv \sum_{\sigma=1}^4 \vartheta_\sigma \mathcal{D}^{(\sigma)}(\check{z}) \equiv \vartheta_1 \mathcal{D}^{(1)}(\check{z}) + \vartheta_2 \mathcal{D}^{(2)}(\check{z}) + \vartheta_3 \mathcal{D}^{(3)}(\check{z}) + \vartheta_4 \mathcal{D}^{(4)}(\check{z}) = 0, \tag{29}$$

where $\mathcal{D}^{(\sigma)}(\check{z})$, $\sigma = 1, 2, 3, 4$, are four linearly independent quadrics of P_3 , and where the parameters ϑ_σ run through all the real numbers. Next, consider the quadratic transformation

$$q\check{z}'_\sigma = \mathcal{D}^{(\sigma)}(\check{z}), \quad q \neq 0, \tag{30}$$

where \check{z}'_σ are regarded as the homogeneous coordinates of a point in a *second* projective space, P'_3 . Combining the last equations and Eq. (29), we get

$$\vartheta_1 \check{z}'_1 + \vartheta_2 \check{z}'_2 + \vartheta_3 \check{z}'_3 + \vartheta_4 \check{z}'_4 = 0. \tag{31}$$

The comparison of this equation with Eq. (16) implies that the *quadrics* of web (29) in P_3 are correlated by the above-given transformation with the *planes* in P'_3 . The quadrics $\mathcal{D}_\vartheta(\check{z})$ may have *base* points, i.e. the points shared by them all. Since each such point is a common zero of $\mathcal{D}^{(1)}(\check{z})$, $\mathcal{D}^{(2)}(\check{z})$, $\mathcal{D}^{(3)}(\check{z})$ and $\mathcal{D}^{(4)}(\check{z})$, it renders Eq. (30) illusory; that is, the base points of the web have *no* counterparts in P'_3 .

Now, let us take three different, linearly independent planes in P'_3 . These define a *unique* point, viz. their meet. The corresponding three distinct quadrics in P_3 have, however, $(2 \times 2 \times 2 =)8$, not all necessarily real and/or distinct, points of intersection in common. So, transformation (29) establishes, in general, an eight-to-one correspondence between the points of P_3 and P'_3 . It may, however, happen that seven points of the eight are the base points of the web; then, obviously, the transformation is a one-to-one type, or birational. Such a web is called *homaloidal*, and the corresponding transformation – a *Cremona* transformation [30]. In other words, given a web of quadrics, Eq. (29), the necessary and sufficient condition for a Cremona transformation to exist is that *any three* quadrics of the web have one and only one point of intersection that varies with the parameters ϑ_σ . In this case, the reverse transformation from P'_3 to P_3 is also unambiguous and can be written as

$$q\check{z}_\sigma = \mathcal{P}^{(\sigma)}(\check{z}'), \quad q \neq 0, \tag{32}$$

where $\mathcal{P}^{(\sigma)}(\check{z}')$ are homogeneous polynomials whose degree is *not* necessarily equal to 2. Then,

$$\mathcal{P}_\eta(\check{z}') \equiv \sum_{\sigma=1}^4 \eta_\sigma \mathcal{P}^{(\sigma)}(\check{z}') \equiv \eta_1 \mathcal{P}^{(1)}(\check{z}') + \eta_2 \mathcal{P}^{(2)}(\check{z}') + \eta_3 \mathcal{P}^{(3)}(\check{z}') + \eta_4 \mathcal{P}^{(4)}(\check{z}') = 0, \tag{33}$$

the η_σ 's being real-valued parameters, represents clearly a homaloidal web in P'_3 whose surfaces correspond to the planes of P_3 .

To find the degree of $\mathcal{P}_\eta(\check{z}')$, we proceed as follows. We first define the order of a curve of the projective space as the number of points which the curve shares with an arbitrary plane, and introduce without proof that the intersection of two surfaces of degree m and n is a curve of order mn . Then, we consider two distinct quadrics of (29). If the base (i.e. shared by all the quadrics) curve of this web, \mathcal{C}_b , is of order g , the residual curve of intersection of the two quadrics, \mathcal{C}_r , is of order $2 \times 2 - g = 4 - g$. Now, this residual curve is mapped by transformation (30) onto a line of P'_3 ; and this line clearly meets $\mathcal{P}_\eta(\check{z}')$ in the same number of points as \mathcal{C}_r meets a generic plane of P_3 , which equals $4 - g$. Hence, the inverse transformation, Eq. (32), will be of the same degree as the original one, Eq. (30), only if the base curve is a second degree curve, i.e. a conic; if \mathcal{C}_b is a line, the inverse homaloidal web comprises cubic surfaces, and, finally, if \mathcal{C}_b reduces to a point, the inverse surfaces are of the fourth degree (quartics).

3.3. Spacetime as a fundamental manifold of a Cremona transformation

We have found that each homaloidal web contains anomalous elements, viz. the base elements, which make the corresponding Cremona transformations illusive. There also exists a different, and much more important for our further purposes, kind of exceptional elements: these, called *fundamental* elements, are defined as the elements in one projective space that corresponds to the base points of the associated homaloidal web in the other space. The totality of the fundamental elements form a manifold whose underlying algebro-geometrical structure, in its pencil understanding, is assumed to mimic sufficiently well that of the observed macro-spacetime. As there are two fundamental manifolds, one in either projective space, we have to be more specific: namely, we *take spacetime to be represented by the fundamental manifold of P_3 , i.e. by the manifold associated with the web of quadrics*, Eq. (29). With this assumption, we are definitely carried beyond the realm of coplanarity and, as we shall see in what follows, arrive at a remarkably fertile ground on which our pencil view of spacetime acquires a qualitatively new footing and considerably sharpens its predictive power.

3.4. Quadro-cubic Cremona transformations and the observed dimensionality and signature of spacetime

To demonstrate this, we shall in the sequel focus our attention on the Cremona transformations associated with a web of quadrics possessing a base *line*. As the inverse web consists, as shown above, of cubic surfaces, these Cremona transformations are usually called the *quadro-cubic* ones [30].

In order for the quadrics' web to be homaloidal, it must feature, in addition to the base line, \mathcal{L}^B , a triple of base *points*, B_i , $i = 1, 2, 3$ [30]; the latter are regarded here as real, distinct, not lying on a line and none of them being incident with the base line. Under these assumptions, the homogeneous coordinate system can be chosen so that

$$\mathcal{L}^B : \check{z}_1 = 0 = \check{z}_2, \tag{34}$$

$$B_1 : \varrho \check{z}_x = (1, 0, 0, 0), \quad \varrho \neq 0, \tag{35}$$

$$B_2 : \varrho \check{z}_x = (0, 1, 0, 0), \quad \varrho \neq 0, \tag{36}$$

$$B_3 : \varrho \check{z}_x = (1, 1, 1, 0), \quad \varrho \neq 0. \tag{37}$$

The corresponding web of quadrics is thus of the form

$$\mathcal{D}_\varrho^*(\check{z}) = \vartheta_1 \check{z}_1(\check{z}_3 - \check{z}_2) + \vartheta_2 \check{z}_2(\check{z}_3 - \check{z}_1) + \vartheta_3 \check{z}_1 \check{z}_4 + \vartheta_4 \check{z}_2 \check{z}_4 = 0, \tag{38}$$

for each of Eqs. (34)–(37) makes, indeed, the last equation vanish identically. This web generates the following Cremona transformation ($\varrho \neq 0$):

$$\varrho \check{z}'_1 = \check{z}_1(\check{z}_3 - \check{z}_2), \tag{39}$$

$$\varrho \check{z}'_2 = \check{z}_2(\check{z}_3 - \check{z}_1), \tag{40}$$

$$\varrho \check{z}'_3 = \check{z}_1 \check{z}_4, \tag{41}$$

$$\varrho \check{z}'_4 = \check{z}_2 \check{z}_4. \tag{42}$$

The inverse transformation, as it can straightforwardly be verified, is of a third-order, namely ($\varepsilon \neq 0$):

$$\varepsilon \check{z}_1 = \check{z}'_3 \left(\check{z}'_1 \check{z}'_4 - \check{z}'_2 \check{z}'_3 \right), \tag{43}$$

$$\varepsilon \check{z}_2 = \check{z}'_4 \left(\check{z}'_1 \check{z}'_4 - \check{z}'_2 \check{z}'_3 \right), \tag{44}$$

$$\varepsilon\check{z}_3 = \check{z}'_3\check{z}'_4(\check{z}'_1 - \check{z}'_2), \tag{45}$$

$$\varepsilon\check{z}_4 = \check{z}'_3\check{z}'_4(\check{z}'_3 - \check{z}'_4), \tag{46}$$

and it relates to the following homaloidal web of cubic surfaces:

$$\begin{aligned} \mathcal{D}'_\eta(\check{z}') &= \eta_1\check{z}'_3(\check{z}'_1\check{z}'_4 - \check{z}'_2\check{z}'_3) + \eta_2\check{z}'_4(\check{z}'_1\check{z}'_4 - \check{z}'_2\check{z}'_3) \\ &+ \eta_3\check{z}'_3\check{z}'_4(\check{z}'_1 - \check{z}'_2) + \eta_4\check{z}'_3\check{z}'_4(\check{z}'_3 - \check{z}'_4) = 0. \end{aligned} \tag{47}$$

The base manifold of this aggregate comprises four distinct lines:

$$\mathcal{L}'_D : \check{z}'_3 = 0 = \check{z}'_4, \tag{48}$$

$$\mathcal{L}'_1 : \check{z}'_2 = 0 = \check{z}'_4, \tag{49}$$

$$\mathcal{L}'_2 : \check{z}'_1 = 0 = \check{z}'_3, \tag{50}$$

$$\mathcal{L}'_3 : \check{z}'_1 - \check{z}'_2 = 0 = \check{z}'_3 - \check{z}'_4, \tag{51}$$

the first one being the *double* line of every cubic in (47). To see this, we consider a generic plane passing through \mathcal{L}'_D :

$$\check{z}'_3 = \kappa\check{z}'_4 \tag{52}$$

with κ running through all the real numbers plus infinity. Inserting this equation into Eq. (47) yields

$$0 = \check{z}'_4^2 \left[(\eta_1\kappa + \eta_2 + \eta_3\kappa)\check{z}'_1 - \kappa(\eta_1\kappa + \eta_2 + \eta_3)\check{z}'_2 + \eta_4\kappa(\kappa - 1)\check{z}'_4 \right], \tag{53}$$

from where it is apparent that every plane through \mathcal{L}'_D meets the cubic $\mathcal{D}'_\eta(\check{z}')$ in this line counted twice, and in another line,

$$\check{z}'_3 - \kappa\check{z}'_4 = 0 = (\eta_1\kappa + \eta_2 + \eta_3\kappa)\check{z}'_1 - \kappa(\eta_1\kappa + \eta_2 + \eta_3)\check{z}'_2 + \eta_4\kappa(\kappa - 1)\check{z}'_4. \tag{54}$$

We now proceed to examine the structure of the *fundamental* manifold associated with the web of quadrics. We find that Eqs. (39)–(42) send the plane $\check{z}_2 = 0$ into the line \mathcal{L}'_1 , $\check{z}_1 = 0$ into \mathcal{L}'_2 , $\check{z}_1 - \check{z}_2 = 0$ into \mathcal{L}'_3 , and, finally, $\check{z}_4 = 0$ into \mathcal{L}'_D . This means that the fundamental elements of P_3 are located in these *four* planes only. The four planes are, however, not all equivalent: the first three are linearly dependent, sharing the line \mathcal{L}^B (and containing, respectively, the points B_1 , B_2 and B_3), whilst the remaining plane, viz. the $B_1B_2B_3$ one, is left aside – see Fig. 3, *left*. But there is much more (intricacy) to this remarkable *three-to-one* factorization than meets the eye. This is revealed when the character of the fundamental elements in each of the planes is concerned.

To this end in view, let us consider a pencil of lines in the plane $B_1\mathcal{L}^B$ whose vortex is the point B_1 . As the variable point of \mathcal{L}^B can be parametrized as

$$\mathcal{L}^B(\check{z}) : \varrho\check{z}_x(\vartheta) = (0, 0, \vartheta, 1), \quad \varrho \neq 0, \tag{55}$$

the moveable point of a line \mathcal{L}_ϑ in the pencil in question is obviously given by

$$\mathcal{L}_\vartheta(\check{z}) : \varrho\check{z}_x(\kappa) = (1, 0, 0, 0) + \kappa(0, 0, \vartheta, 1) = (1, 0, \kappa\vartheta, \kappa), \quad \varrho \neq 0, \tag{56}$$

where κ , like ϑ , runs through all the reals and infinity. Putting the last equation into Eqs. (39)–(42), we obtain

$$\tilde{\varrho}\check{z}'_x = (\kappa\vartheta, 0, \kappa, 0) = \kappa(\vartheta, 0, 1, 0), \quad \tilde{\varrho} \neq 0, \tag{57}$$

and immediately see that, for a fixed ϑ , the last equation represents one and the same point of P'_3 irrespective of the value of κ ; that is, a(ny) line of the pencil transforms as *a whole* into a single point

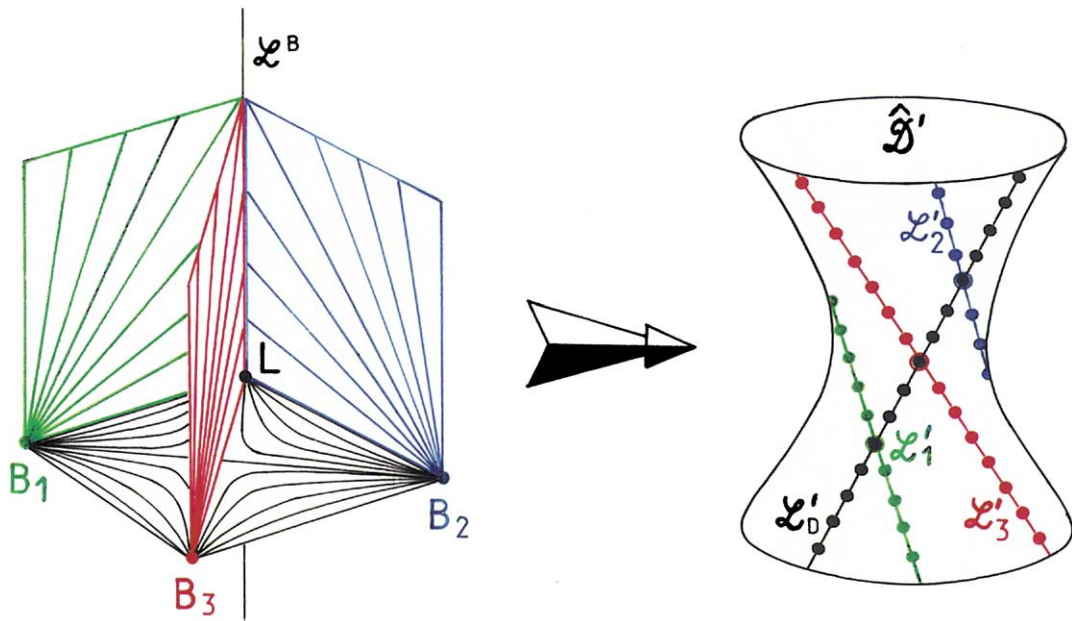


Fig. 3. Left: The fundamental manifold of P_3 associated with the homaloidal web of quadrics defined by Eq. (38). Right: Its P'_3 image furnished by the corresponding Cremona transformation, Eqs. (39)–(42). The manifold comprises three pencils of lines and a single pencil of conics, and its image features, respectively, three skew lines and a fourth line incident with each of the three. The symbols and notation are given in the text.

of \mathcal{L}'_1 , and is thus a *fundamental line* of P_3 . This mapping is clearly of a one-to-one type, i.e. to two distinct lines $\mathcal{L}_{\vartheta^{(1)}}$ and $\mathcal{L}_{\vartheta^{(2)}}$, $\vartheta^{(1)} \neq \vartheta^{(2)}$, there correspond two different points of \mathcal{L}'_1 . Performing the same reasoning for the $i = 2$ and $i = 3$ cases, we find a completely analogous result: all in all, given the plane $B_i\mathcal{L}^B$, $i = 1, 2, 3$, the lines of the pencil centred at B_i are the fundamental lines of P_3 , being transformed by Eqs. (39)–(42) into the points of P'_3 lying on the base line \mathcal{L}'_i , $i = 1, 2, 3$, of the web of cubics, Eq. (47) – as depicted in Fig. 3. And what about the remaining ‘base’ plane, the $\check{z}_4 = 0$ one? Here we find, as expected, a different situation, because this plane cuts the web of quadrics in a pencil of *conics*, viz.

$$\mathcal{Q}_{\vartheta}(\check{z}) = \vartheta_1\check{z}_1(\check{z}_3 - \check{z}_2) + \vartheta_2\check{z}_2(\check{z}_3 - \check{z}_1) = 0, \tag{58}$$

the conics having in common the three base points B_i , $i = 1, 2, 3$, and the point L at which the plane meets the base line \mathcal{L}^B – see Fig. 3, *left*. From the last equation and Eqs. (39)–(42) we find that the *whole* conic $\mathcal{Q}_{\vartheta \equiv \vartheta_2/\vartheta_1}$ is mapped into a single point of P'_3 :

$$q\check{z}'_4 = (-\vartheta, 1, 0, 0), \quad q \neq 0, \tag{59}$$

which belongs to the double base line \mathcal{L}'_D ; hence, also every conic of pencil (58) is a *fundamental curve* of P_3 . Again, the correspondence has a one-to-one character.

At this point it should be fairly obvious why and how the already enunciated identification of pencil-spacetime with the fundamental manifold corresponding to the homaloidal web of quadrics defined by Eq. (38) unravels, at one stroke, two long-standing mysteries connected with the macroscopic structure of spacetime, viz. its dimensionality and signature. *The macro-dimensionality of spacetime amounts to four simply because we have four pencils of fundamental elements, and its 3 + 1 signature reflects nothing but the fact that three of them are composed of lines (spatial dimensions), whereas the remaining one features conics as the fundamental constituents (time).* It is so natural and so amazingly simple an explanation, following up nicely with the P_2 model and striking closest to the heart of the matter in our current state of understanding! An appeal of this spatial model is further

substantiated when we notice that *it uniquely specifies the type of a pencil of conics* (Eq. (58)), *i.e. the global character of the time dimension* – a property which had to be independently postulated in our debut model (see Section 2.3).

In order to introduce a non-trivial, arrow-resembling structurization into the pencil of fundamental conics, Eq. (58), we have, as in the planar case, to dehomogenize the projective space. This can be done in a way that is a natural and straightforward extension of what we did in the projective plane. To refresh our memory (Section 2.2), we picked up one line that was pronounced to have a special standing with respect to the other lines of P_2 . Since the spatial analogue of a line is obviously a plane, we pursue the same strategy: we choose one plane of P_3 , we may call the d -plane, and assign to it a special status. If this plane does not coincide with the plane of pencil (58), the $B_1B_2B_3$ plane, it cuts the latter in the unique line. This line clearly plays the role of the d -line in the plane in question and, so, the rest of our analysis concerning time’s arrow almost completely reduces to that carried out in our strictly planar model, Sections 2.2 and 2.3; the only exception being that the pencil of conics defined by Eq. (58) is of a different type than that given by Eq. (6).

3.5. Cremona transformations: a mediating link between physics and psychology?

The fundamental manifold of P_3 has just been found to be a remarkably useful qualitative framework for the basic number-geometrical characteristics of ‘sensual’ macro-spacetime. Yet, it is also its P'_3 image, *viz.* the base manifold of the inverse web of cubics, that should play in our theory a part no less prominent than the former. Then, the natural question arises: what kind of spacetime does this second manifold correspond to?

To properly address this question, we shall have to carry out a bit closer inspection of the configuration of four base lines defined by Eqs. (48)–(51). As already shown, one of the lines, \mathcal{L}'_D , differs from the others, \mathcal{L}'_i , $i = 1, 2, 3$, in being the double line of a cubic $\mathcal{P}'_\eta(z')$. This distinction becomes more marked when we notice that all the four lines lie on the ruled quadric

$$\widehat{\mathcal{Q}}'(z') = z'_1z'_4 - z'_2z'_3 = 0. \tag{60}$$

As this equation is formally identical to Eq. (26), conceiving Eqs. (27) and (28) as primed we find that \mathcal{L}'_i , $i = 1, 2, 3$, are contained in the μ -system (corresponding, respectively, to $\mu = \infty, 0, 1$), whereas \mathcal{L}'_D is of the λ -family ($\lambda = \infty$) – as sketched in Fig. 3, *right*. Hence, a three-to-one splitting is also an inherent feature among the elements of the base manifold of P'_3 . In the present case, however, it is not pronounced so well as in the case of the fundamental manifold of P_3 , for a projective line is a much simpler structure than a projective plane. So, after relating the fundamental manifold of P_3 with the spacetime as experienced/perceived by our senses, we are daringly led to identify the base manifold of P'_3 with the spacetime as described by physics. A principal justification for such a claim goes as follows. From a physicist’s point of view, there is no distinction between time and space as far as their internal structure is concerned; the only difference between the two is embodied in the Lorentz signature of a metric tensor on an underlying differentiable manifold. And a strikingly similar phenomenon is taking place on the base manifold of P'_3 . Here, there is also no structural difference between the time dimension and spatial coordinates, for all of them are represented by (base) *lines*; yet, the time coordinate, represented by \mathcal{L}'_D , acquires a different footing than the three dimensions of space, generated by lines \mathcal{L}'_i , $i = 1, 2, 3$, simply because \mathcal{L}'_D belongs to a different regulus of $\widehat{\mathcal{Q}}'$ than the triple of lines $\mathcal{L}'_1\text{--}\mathcal{L}'_3$.

We thus have at our disposal two unequivalent, yet robust on their own, ‘Cremona’ ways of picturing the universe, both reproducing strikingly well its dimensionality and signature: one (the fundamental manifold of P_3 ; Fig. 3, *left*) relates more to the properties of the universe as presented through our sensory processes (a ‘subjective’ view), whilst the other (the base manifold of P'_3 ; Fig. 3, *right*) portrays rather its physical content (an ‘objective’ view). And the two representations are intimately coupled to each other via a specific transformation, *viz.* the generic quadro-cubic Cremona transformation defined by Eqs. (39)–(42). Being algebraically elegant and geometrically very simple, this transformation may thus offer extraordinary promise for being a crucial stepping-stone towards bridging the gap between two seemingly irreconcilable domains of the human inquiry, *viz.* physics and psychology.

3.6. The dimensionality of space and the global structure of time

The attentive reader surely noticed that the web of quadrics defined by Eq. (38) represents the *most* general case of a homaloidal web of quadrics featuring a real base line, as the residual base points B_i , $i = 1, 2, 3$, are all real and distinct. We shall, therefore, call the spacetime(s) associated with this web the *generic* quadro-cubic Cremona spacetimes. The purpose of this section is to briefly examine a couple of specific quadro-cubic configurations, namely those arising when two of the base points coalesce and/or transform into a conjugate complex point-pair, and see the difference in the dimensionality/structure of the corresponding ‘fundamental’ spacetimes when compared with the generic case.

We shall deal first with the case that we get from the generic one by making two base points, say B_2 and B_3 , approach each other until they ultimately get united, the merger point being denoted as B . Clearly, the corresponding planes of fundamental lines, $B_2\mathcal{L}^B$ and $B_3\mathcal{L}^B$, fuse in the limit into a single plane, $B\mathcal{L}^B$, and the corresponding pencil-space thus becomes *two*-dimensional only. This reduction in the dimensionality of space is accompanied by a change in the structure of the time dimension. Employing the classification of conics’ pencils given in [16], we see that the resulting pencil of fundamental conics, since displaying only three distinct real base points L , B_1 and B , is of the $3\%2$ -type, while our generic pencil, Eq. (58), belongs to the $4\%3$ -class. When two of the base points, we shall again denote as B_2 and B_3 , become imaginary (conjugate complex), there remains only one real plane containing a pencil of fundamental lines, viz. the $B_1\mathcal{L}^B$ plane, and, hence, space is endowed with a *single* dimension. The corresponding pencil of fundamental conics possesses only two real base points, L and B_1 , and is of type $2\%1$ -B [16].

These two special cases illustrate sufficiently well the fact that the number of observed spatial coordinates and the global structure of time dimension are closely related with each other characteristics of our quadro-cubic pencil-spacetimes. This property ties in very nicely with the relationship ‘arrow of time \leftrightarrow three-dimensionality of space’ found in the planar model (Section 2.3), and may well turn out to be a valuable hint for superstring and supermembrane theorists in their quest for answering the question why the alleged extra dimensions of spacetime have to be spontaneously compactified (see, e.g., Ref. [31]).

3.7. A relationship with Cantorian spacetime

Following up its two-dimensional predecessor, the generic quadro-cubic Cremona spacetime sitting in the projective space is also seen to share some formal features with the Cantorian space, $\mathcal{E}^{(\infty)}$, whose principal properties were highlighted in the introductory section. Since any pencil of lines/conics in P_3 may be regarded as a potential spatial/temporal dimension, the Cremona spacetime, like $\mathcal{E}^{(\infty)}$, is formally *infinite*-dimensional. Hence, both the manifolds must undergo a specific dimensional reduction in order to yield the finite number of dimensions offered to our senses. In the present case, this reduction is of algebro-geometrical nature, for we demand the structure of the ‘residual’ macro-manifold to be in conformity with particular algebraic transformations, while in the case of $\mathcal{E}^{(\infty)}$, it results from a sort of statistical averaging, i.e. it has a number-probabilistic character [7–13].

A second intriguing feature making the two models akin to each other concerns the signature of spacetime. Within the $\mathcal{E}^{(\infty)}$ framework, the difference between space and time, as already mentioned in the introductory section, stems from the two different concepts of effective dimensionality, viz. topological and Hausdorff. The topological dimension, being a *simpler* concept of the two, is argued to grasp only spatial degrees of freedom, while the Hausdorff one, being more complex, is understood to incorporate also the time dimension [7]. In our model, we have two distinct kinds of pencils instead. If we exclusively confine to those consisting of lines, which are *simpler* ones, we can get hold just of the spatial manifold. Time enters only if we also consider pencils of conics.

4. Summarizing conclusion

We gave a lucid exposition of a very interesting and fruitful generalization of the pencil concept of spacetime by simply raising the dimensionality of its projective setting from two to three. When

compared with its two-dimensional sibling, this extended, three-dimensional framework brings much fresh air into old pressing issues concerning the structure of spacetime, and allows us to look at the latter in novel, in some cases completely unexpected ways. Firstly, and of greatest importance, this framework offers a natural qualitative elucidation of the observed dimensionality and signature of macro-spacetime, based on the sound algebro-geometrical principles (Section 3.4). Secondly, it sheds substantial light at and provides us with a promising conceptual basis for the eventual reconciliation between the two extreme views of spacetime, namely physical and perceptual (Section 3.5). Thirdly, it gives a significant boost to the idea already indicated by the planar model that the multiplicity of spatial dimensions and the generic structure of time are intimately linked to each other (Section 3.6). Finally, being found to be formally on a similar philosophical track as the fractal Cantorian approach, it grants the latter further credibility (Section 3.7).

Acknowledgements

I am very grateful to Mr. Pavol Bendík for careful drawing of the figures and thank Dr. Rosolino Buccheri (CNR, Palermo) for raising many intriguing conceptual questions related to the physical meaning of quadro-cubic Cremona transformations.

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