Title of the "Research in Pairs" Programme
Finite Geometries Behind the Black–Hole–Qubit Correspondence

MSC2010: 51Axx; 51Cxx; 51Exx; 81P45; 81R05; 81T30; 83E30
IMU1995: 13 Mathematical Physics

Dates
February 24 – March 16, 2013

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Background Settings, Specific Goals and Activity Focus
Recently, striking multiple relations have been established between the physics of stringy black hole solutions and the entanglement of qubits in quantum information theory [1–20]. Though this black-hole-qubit correspondence (BHQC) still begs for a firm physical basis, the underlying relations have repeatedly proved to be useful for obtaining new insights into one of these fields by exploiting the methods established within the other. The main unifying theme in these papers is the correspondence between the Bekenstein-Hawking entropy formula [21,22] for black hole and black string solutions in $D = 4$ and $D = 5$ supergravities arising from string/M-theory compactifications and certain entanglement invariants of multiqubit/multiqutrit systems. As a new unifying agent in some of these papers [5,12,13] the role of finite geometric ideas have been emphasized. In particular, it has been shown [4,5] that the smallest projective plane, the Fano plane with seven points and seven lines, with points conveniently labeled by seven 3-qubit states can be used to describe the structure of the E_{7}-symmetric black hole entropy formula of $N = 8$, $D = 4$ supergravity. Moreover,

this geometric representation based on the fundamental 56-dimensional representation of E_{7} in terms 28 electric and 28 magnetic charges enabled a diagrammatic understanding of the consistent truncations with 32, 24 and 8 charges as a restriction to quadrangles, lines and points of this plane [5]. Though the Fano plane turned out to be a crucial ingredient also in later studies, this geometric representation based on the tripartite entanglement of seven qubits presents formidable theoretical and experimental challenges [10].

To circumvent these problems, an attempt was made in [12] to construct a new representation using merely three-qubits. The basic idea was to use the central quotient of the 3-qubit Pauli group [23], well known from studies concerning quantum error correcting codes. This Abelian group can be described by the 63 real operators of the Pauli group with multiplication up to a sign. These 63 operators can be mapped bijectively to the points of another remarkable finite geometrical object called the split Cayley hexagon of order two having 63 points and 63 lines, with a subgeometry (the complement of one of its geometric hyperplanes) being the Coxeter graph with 28 points/vertices. This graph has been related to the charge configurations of the E_{7} symmetric black hole entropy formula [12]. The advantage of this representation was a clear understanding of an automorphism of order 7 relating the seven STU subsectors of N = 8, D = 4 supergravity and the explicit appearance of a discrete PSL(2,7) symmetry of the entropy formula. The permutation symmetry of the STU model (triality) in this picture arises as a subgroup of PSL(2,7).

Motivated by these findings, Lévay *et al.* have next [13] succeeded in showing that the $E_{6(6)}$-symmetric entropy formula describing black holes and black strings in D = 5 is intimately tied to the geometry of the generalized quadrangle GQ(2,4) with automorphism group the Weyl group W(E_{6}). The 27 charges correspond to the points and the 45 terms in the entropy formula to the lines of GQ(2,4). Different truncations with 15, 11 and 9 charges are represented by three distinguished subconfigurations of GQ(2, 4), well known to finite geometers; these are the “doily” [i.e. GQ(2, 2)] with 15, the “perp set” of a point with 11, and the “grid” [i.e. GQ(2, 1)] with nine points, respectively. In order to obtain the correct signs for the terms in the entropy formula, we used a noncommutative labeling for the points of GQ(2, 4). For the 40 different possible truncations with nine charges this labeling yields 120 Mermin squares—objects well known from studies concerning Bell-Kochen-Specker-like theorems. These results are intricately connected to our previous ones obtained for the E_{7}-symmetric entropy formula in D = 4 by observing that the structure of GQ(2,4) is linked to a particular kind of geometric hyperplane of the split Cayley hexagon of order 2, featuring 27 points located on nine pairwise disjoint lines (a distance-3-spread). We also conjectured that the different possibilities of describing the D = 5 entropy formula using Jordan algebras, qubits and/or qutrits correspond to employing different coordinates for an underlying noncommutative geometric structure based on GQ(2,4).

The main objective of the project is to examine in detail other known stringy black hole solutions with a view of revealing some other finite geometries behind the scene. To this end, one has to realize at the very beginning that, unlike the Fano plane, neither the split Cayley hexagon of order two, H_{2}, nor the generalized quadrangle GQ(2,4) are self-dual point-line incidence structures, i. e., neither of them is isomorphic to the dual obtained by swapping the roles of points and lines, H_{2}^{D} and GQ(4,2). These duals, surprisingly, have not been linked with any known entropy formulas. Why? We believe that answering this question is a key element for meeting the project’s objective. There are, obviously, two possibilities: there do exist yet-to-be-discovered black hole/string solutions backed by these two duals and we can predict their basic properties, or no such solutions exist. Be it this way or that, this task will necessitate to properly understand the finest traits of the difference between H_{2} and H_{2}^{D}, and between GQ(2,4) and GQ(4,2) as well. Here, we shall focus on comparing different embeddings of the geometries in question, their geometric hyperplanes and their complements as well as their Veldkamp spaces; we already found that the Veldkamp space of GQ(2,4) is isomorphic to PG(5,2), whilst that of GQ(4,2) is even not a linear space.

Successful accomplishment of the above-outlined task should give us clear clues as to where to look for further promising finite geometric candidates of the BHQC. In addition to generalized polygons, symplectic and orthogonal polar spaces and their duals, we also aim at examining Hermitian varieties $H(d, q^{2})$ for certain specific values of dimension d and order q. A particularly interesting case is d=2, where a Hermitian curve is a representative of a broader class of finite geometries called unitals (i. e., 2-$(q^{3}+1, q+1, 1)$-designs); here, already for q=3 we have a large variety of unitals of a different degree of symmetry, some of which are even embeddable into non-Desarguessian projective planes of order nine. Given the fact [14] that the structure of extremal stationary spherically symmetric black hole solutions in the STU model of D=4, N=2 supergravity can be described in terms of four-qubit systems, the H(3, 4) variety is also notable, because its points can be identified with the images of triples of mutually commuting operators of the generalized Pauli group of four-qubits via a geometric spread of lines of PG(7,2) [24]. In this regard, we would also like to have a closer look at (the spin-embedding of) the dual polar space DW(5, 2) (into PG(7, 2)), since the points of this space are in a bijective correspondence with the points of a hyperbolic quadric $Q^{+}(7,2)$ and, so, with the set of symmetric operators of the real four-qubit Pauli group [24,25].

The next step will likely be the one into the realm of finite geometries that are not linear spaces. This is already indicated by our preliminary analysis of the I_4 invariant in the N=8 black hole/qubit correspondence [10]. Identifying the 56 charges with the points and the terms in the invariant formula with the lines of an alleged point-line incidence structure, such a geometry features instances of two or more lines passing through two distinct points – a clear violation of the linearity of space. This means that, at least in this particular case, we have to pass to geometries defined over rings which are not fields. This also seems to be a promising avenue as such geometries have already been successfully applied in QIT (see, e.g., [26,27]). Here, we meet a class of particularly appealing geometries defined over rings of ternions, as these feature free cyclic submodules (that is, points) of two qualitatively different kinds [28]: those generated by unimodular vectors and having counterparts in field geometries, as well as those generated by non-unimodular vectors, which have no analogue in any geometry defined over a field. It is worth mentioning here that such "non-unimodular" part of the projective plane defined over the smallest ring of ternions, dubbed the "Fano-snowflake" [29], features the ordinary Fano plane as the core geometry.

There is also an infinite family of tilde geometries associated with non-split extensions of symplectic groups over a Galois field of two elements that are worth a careful look at. One of the simplest of them, $\tilde{W(2)}$, is the flag-transitive, connected triple cover of the unique generalized quadrangle GQ(2,2) (the "doily") that describes the commutation properties of the two-qubit Pauli group [27]. $\tilde{W(2)}$ is remarkable in that it can be, like H_{2} and GQ(2,4), embedded into PG(5, 2). This embedding goes as follows. Let S be a regular line spread of PG(5,2), i. e., a set of 21 pairwise disjoint lines of PG(5,2) such that the PG(3,2)-space generated by any two members of S contains five elements of S. Note that S endowed with these 5-sets is a copy of PG(2,4). Let T \subset S be a set of six lines with the property that each three of them generate PG(5,2). (This 6-set is the image of a hyperoval in PG(2, 4)). Then the 45 points of PG(5,2) not incident with any element of T, together with the lines of PG(5, 2) which are contained in PG(3, 2)s generated by two elements of T that do not belong to S and do not meet any member of T, define an embedding of \tilde{W(2)}. From this construction it is obvious that there exists an intricate link between \tilde{W(2)}, three particular kinds of geometric hyperplanes of H_{2} [30] and a Hermitian spread of GQ(2,4). Hence, more complex tilde geometries certainly deserve our attention.

The next aspect of the BHQC we plan to tackle is graph theoretical. This aspect is very closely related to the above-discussed finite geometrical one because both GQ(2,2) and a pair of dual to each other generalized hexagons of order two are bislim geometries, i. e. geometries where each line has three points and each point is on three lines, and in any such geometry the complement of a geometric hyperplane represents a cubic graph. A cubic graph is one in which every vertex has three neighbours and so, by Vizing's theorem, three or four colours are required for a proper edge colouring of any such graph; and there, indeed, exists a very interesting but somewhat mysterious family of cubic graphs, called snarks, that are not 3-edge-colourable, i.e. they need *four* colours [31,32]. To avoid trivial cases, a snark is defined as a cubic cyclically 4-edge-connected graph with chromatic index four whose girth is at least five. The importance of snarks in graph theory lies in their intimate connection with the four-colour theorem (that is equivalent to the statement that every bridgeless cubic planar graph is 3-edge-colourable, hence no snark can be planar) and in the existence of a number of open profound conjectures that have snarks as minimal counterexamples, like the Cycle Double Cover Conjecture or Tutte's 4-flow and 5-flow ones. Why should we be bothered with snarks? Well, because the smallest of all snarks, the Petersen graph, is isomorphic to the complement of a particular kind of hyperplane (namely an ovoid) of GQ(2,2)! There are only three distinct kinds of hyperplanes in GQ(2,2) [27], but as many as 25 in the split Cayley hexagon of order two and as many as 14 in its dual [30]. So it is very likely that the complements of some of them are snarks and it is desirable to see if this holds true and, if so, what the properties of these snarks are. If we do find some snarks here, or in any other BHQC-relevant bislim geometry, this could have at least two-fold bearing on our understanding of the BHQC. On the one hand, there exists a noteworthy built-up principle of creating snarks from smaller ones embodied in the (iterated) dot product operation on two (or more) cubic graphs [33,34]; given arbitrary two snarks, their dot product is always a snark. In fact, a majority of known snarks can be built this way from the Petersen graph alone. Hence, the Petersen graph is an important "building block" of snarks; in this light, it is not so surprising to see GQ(2,2) playing a similar role in QIT. And it is not unlikely that a similar built-up principle works also in the context of the BHQC. On the other hand, the nonplanarity of snarks immediately poses a question on what surface a given snark can be drawn without crossings, i. e. what its genus is [35,36]. The Petersen graph can be embedded on a torus and, so, is of genus one. If other snarks emerge in the context of the BHQC, comparing their genera with those of manifolds occurring in major compactifications of string theory will also be an insightful task.

The final aspect of our prospective research is closely tied to an intriguing observation by Eguchi, Ooguri and Tachikawa [37] that the elliptic genus of K3 surface may be expanded in a linear combination of $N = 4$ superconformal characters and that the coefficients of the non-BPS $N = 4$

characters in the elliptic genus decomposition coincide with the dimensions of some irreducible and reducible representations of the Mathieu group M_{24}. If this conjecture, viz. that the elliptic genus of K3 carries indeed an action of M_{24}, proves to be correct, as some recent works seem to indicate [38, 39], then there are several other prominent geometries that may turn out to be relevant for our BHQC. First, it is the unique Steiner system S(5,8,24): a collection of 759 8-element subsets ("octads") of a 24-element set Ω with the property that any 5-element subset of Ω is contained in precisely one octad, whose full group of automorphisms is nothing but M_{24} [40]. Second, it is a near hexagon E_{2} of order (s,t) = (2,14): the points of this near hexagon are the blocks/octads of the above-mentioned Steiner system, the lines are the sets of three mutually disjoint blocks and incidence is natural one [41]. Third, it is the (already mentioned) projective plane of order four, PG(2,4), since the 759 block of the Steiner system (and, so, the 759 points of our near hexagon) can be represented in terms of geometrical objects living in PG(2, 4), namely: the set of 21 lines, the set of 210 symmetric differences of two distinct lines, the set of 168 hyperovals (i. e., sets of 6 points no three of which are collinear) and the set of 360 Baer subplanes (i.e., Fano planes) [42]. The occurrence of these particular finite geometries is remarkable also by the fact that each of them contains a large number of copies of the generalized quadrangle of order two, GQ(2, 2)! In the Steiner system GQ(2,2)s live as sextets (sets of 6 complementary tetrads such that the union of any two of them is an octad) [40], in the near hexagon as so-called quads [41], and in the projective plane they can be recognized as follows: take a hyperoval, the 15 points off the hyperoval and 15 lines cutting the hyperoval, natural incidence, form a GQ(2,2) [43]. And because in our approach a GQ(2,2) is synonymous with a two-qubit Pauli group, this alongside opens up a fascinating avenue of employing this group to help unraveling/clarifying the role of M_{24} in the context of strings compactified on K3 surfaces.

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Expected Outcomes

The increasing occurrence of finite geometries in quantum physics can be likened to that of groups in the first half of the last century; in much the same way as groups replaced a variety of cumbersome methods and techniques used before and provided an elegant unifying framework for many areas of physics, finite geometries seem to do so at least within quantum information theory. So, we believe that finite geometric and graph theoretical methods may turn out to be important guiding principles in where group theoretic techniques offer still too many degrees of freedom. The correspondence between macroscopic entropy formulas obtained for certain black hole solutions in supergravity theories and multi-qubit and -qutrit entanglement measures used in QIT is, as already emphasized, embodied in the occurrence of similar groups of symmetry in these very different contexts. We have currently no conceptual clue why this is so and we do anticipate that finite geometries will provide us with some. From a more technical point of view, finding new finite geometries and snarks behind BHQC could, on the one side, point out the existence of a whole new class of stringy black hole solutions and, on the other side, tell us something new about the finer structure of finite-dimensional Hilbert spaces and groups associated with them. In the latter case, for example, finite geometries underlying BHQCs could also be helpful in classifying multipartite entanglement for more than three qubits, where it is still not clear what conditions are to be imposed on such a classification; although it is known in principle how to generate invariants to a certain group, the real problem consists in the distillation of those invariants relevant for entanglement-related questions.

The main results of the research project will be published in leading international journals in the fields (such as Physical Review A/D, Journal of Physics A: Mathematical and Theoretical, Finite Fields & Their Applications, Journal of Geometry, Advances in Geometry, and Quantum Information & Computation, to mention a few), presented in various forms at (both national and international) conferences, colloquia and/or topical/brainstorming workshops, and simultaneously posted on the internationally recognized physics, mathematics, computer science, quantitative biology and statistics on-line open-access web archive, http://arxiv.org/, in the subfields of "quant-ph", "astro-ph", "math-ph" and "hep-th", as well as on its French scholarly counterpart, http://hal.archives-ouvertes.fr/.

Prospective Timeline

- A thorough inventory and subsequent in-depth inspection of various aspects of all known stringy black hole solutions, associated entropy formulas and already established aspects of the black-hole/qubit correspondence (BHQC).
- Getting deeper insights into the BHQC mathematical formalisms so far employed; particular focus on Jordan algebras, Freudenthal triple system, superqubits, known classifications of entanglement types, etc.
- Entanglement aspects of the BHQC, classification of black hole solutions and classification of entanglement types of few qubit and qutrit systems. Understanding why and under what conditions the two types of classification problems can be mapped into each other. Studying the structure and representation theory of U-duality groups.
- Identifying the physical nature of solutions transforming according to a particular representation (i.e. momentum and flux multiplets of BPS states in M- and matrix theory).
- Proper understanding of why and how the above-mentioned finite geometries (i.e. the Fano plane, split Cayley hexagon of order two and the generalized quadrangle GQ(2,4)) enter the game and ascertaining, in the last two cases, what could be the role of the dual configurations.
- An extensive search for black-hole entropy formulas admitting finite-geometric representation; particular attention paid to the geometry underlying the I_4 invariant in the N=8 BHQC.
- Analysis of algebraic and combinatorial properties of the newly-established BHQC; their relations to the currently known cases and curiosity to know whether the next move must entail geometries linked with tri- or multi-linear forms.
- Summarizing the main achievements; publication and further dissemination of the core results.

Expertise of the Participants

Dr. Lévay, a co-discoverer of the important role of the Fano plane in the context of the BHQC, focuses on string theory and the geometry of quantum entanglement. Dr. Planat is interested in the finite geometrical and group theoretical aspects of QIT, paying particular attention to quantum gates. Dr. Saniga has a deep knowledge of various kinds of finite geometries and some associated graphs.