Identification du project:

Crucial importance of quantum theory for addressing the most fundamental aspects of reality has invariably been at the forefront of theoretical explorations of most prominent scholars, being firmly established by the Aspect *at al.*'s experiment in 1982. Two measurements described by non-commuting observables are inherently uncertain and this led Einstein, Podolsky and Rosen to question the completeness of quantum theory versus the reality of both the observed physical quantities. Using counterfactual arguments applied to distant experimental set-ups they introduced (and immediately rejected) the notion of underlying wholeness, which shortly after gave rise to the concept of quantum entanglement. Bohr believed that no serious conclusion can be drawn from the comparison of thought experiments dealing with mutually incompatible (i.e., non-commuting) observables and thus practically ignored the paradox, proposing another view/paradigm—quantum complementarity. Since the work of Bohm and Bell, the "puzzles" of quantum theory have mainly been discussed within a discrete variable setting of spin-1/2 particles.

In essence, Bell's theorems imply that either the recursive (counterfactual) reasoning about possible experiments should be abandoned, or non-contextual assumptions (implicit in the EPR locality arguments) are to be challenged, or both. One of the simplest illustrations of quantum "mysteries", which also provides a very economical proof of the Bell-Kochen-Specker theorem, employs a 3×3 array of nine observables characterizing two spin-1/2 particles. The three operators in any row or column of such a square, commonly referred to as the Mermin "magic" square, are mutually commuting, allowing the recursive reasoning to be used, but the algebraic structure of observables contradicts that of their eigenvalues. Making use of the basic facts about quantum complementarity and maximal quantum entanglement for two spin-1/2 particles (or two-qubits in terms of quantum information theory), we have recently demonstrated [1] that the 15×15 multiplication table of the associated four-dimensional (generalized Pauli spin) matrices exhibits a so-far-unnoticed geometrical structure, which can be regarded as three pencils of lines in the projective plane of order two (the Fano plane). These three pencil-configurations, each featuring seven points/observables, share a line (called the reference line), and any line comprises three observables, each being the product of the other two, up to a factor -1, i or $-i$. All the three lines in each pencil carry mutually commuting operators; in one of the pencils, which we call the kernel, the observables on two lines share a base of maximally entangled states. The three operators on any line in each pencil represent a row or column of some of Mermin's "magic" squares, thus revealing an inherent geometrical nature of the latter [2]. In the complement of the kernel, the eight vertices/observables are joined by twelve lines, which form the edges of a cube. The lines between the kernel and the cube are pairwise complementary, which means that each vertex/observable is linked with six others.

In order to get a deeper insight into these intriguing geometrical features we have employed a novel approach based on the concept of *projective ring lines* [3], in particular those defined over the direct product ring GF(2)⊗GF(2)⊗…⊗GF(2), n times, with $n = 2$, 3, and 4 and GF(2) denoting the simplest Galois field. The line over GF(2)⊗GF(2) was found to reproduce nicely all the basic qualitative properties of a Mermin square, while to account for a more intricate geometrical structure of the kernel and the cube, one had to employ the lines corresponding to $n = 3$ and $n = 4$, respectively. Although the latter two geometries provide us with important insights into the structure of these configurations per se, they still fail short in harbouring the correct coupling between them. It is therefore necessary to look for a higher order ring line and/or closely-allied geometry in order to get a complete geometrical picture of two- (and, eventually, also higher-order-)qubit systems.

As an important, closely-related side-issue to be addressed in the project are mutually unbiased bases (MUBs). There are numerous ways for constructing complete sets of MUBs, most of them being based on discrete Fourier analysis in Galois fields and Galois rings, discrete Wigner functions, and generalized Pauli matrices. Recently, we [4] revisited the problem of the construction of MUBs in d dimensions from a polar decomposition of the Lie group SU(2) used in conjunction with the cyclic group C(d). In this direction, we have derived a compact formula for the various MUBs in dimension d = p with p prime. The case of the prime power dimension, d = p**e, with p prime and e an integer greater than 1, is much more involved. It constitutes the starting point (together with the corresponding Galois algebra aspects) of the thesis by O. Albouy. We would like to obtain a compact formula as in the case where d = p with p prime. For this purpose, the strategy to be used is similar to the one employed in group theory for solving the so-called missing label problem. The main advantage to use a SU(2) approach to the construction of MUBs in composite systems (here e-qubits systems) is that intricate and entangled states are quite well described in term of the Wigner-Racah algebra of SU(2).

The project proposed primarily aims at finding the proper projective and/or related ring geometrical setting for a generic system of two spin-1/2 particles. To furnish this task will require not only deepening our familiarity with the fine structure of a large number of higher-order rings lines, but also examining other closely related geometrical concepts. As per the former issue, the lines deserving a particular attention are those defined over finite *non*-commutative rings of small order, because the idealizers of a majority of these rings turned out to be nothing but the (commutative) rings we have already employed in our model. We surmise that it is non-commutative rings of order thirty-two that will provide us with a decisive clue in this respect, in particular those whose idealizers are isomorphic either to the simplest non-commutative ring of order eight or to the unique non-commutative ring of order sixteen devoid of two-sided ideals. Concerning the latter task, we intent to have a closer look at so-called *chain* geometries [5], for these are very intimately related with projective ring lines. Successful accomplishments of these objectives is likely to put our understanding of the behaviour of two (and more) spin-1/2 particles on a qualitatively new footing, entailing profound implications for such areas as quantum coding and cryptography, quantum computing and for the very foundations of quantum mechanics itself.

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