ON SECOND ORDER STATISTICAL MOMENTS IN MHD-TURBULENCE

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Abstract: A brief survey concerning investigations of the magnetic field correlation tensor in the framework of the second order correlation approximation is given. A special relation connecting the mean square of the fluctuating magnetic field and the square of the mean magnetic field is applied to the fields observed at the Sun's surface.

Investigating the magnetohydrodynamics of turbulently moving electrically conducting fluids the theory of the statistical moments of first order, the mean fields, was developed. Its application was successful especially with respect to the explanation of the origin of cosmic magnetic fields. More specified to the subject of this meeting the origin of the solar cycle could be explained with some of its basic properties: the periodicity, the symmetry and the butterfly diagram.

However, many observational facts are still unexplained. In some cases it is clear that they are beyond the scope of mean-field magnetohydrodynamics, pertaining to the theory of statistical moments of higher order.

Obviously general investigations of statistical moments of higher order open a wide field. As a first step in this direction we studied (Bräuer and Krause, 1973, 1974) some questions concerning the mean square of the fluctuating magnetic field, i.e. the turbulent magnetic energy.

The magnetic field, $\mathbf{B}$, may be understood as a sum of its mean part, $\bar{\mathbf{B}}$, and its fluctuating part, $\mathbf{B}'$. The same denotation is used for the velocity field, $\mathbf{u}=\bar{\mathbf{u}}+\mathbf{u}'$. We define the correlation tensors of second rank

$$Q_{ij}(\mathbf{x}, \xi, t, \tau)=\overline{u_i'(\mathbf{x}, t)\, u_j'(\mathbf{x}\times\xi, t+\tau)}\,, \tag{1}$$

$$B_{ij}(\mathbf{x}, \xi, t, \tau)=\overline{B_i'(\mathbf{x}, t)\, B_j'(\mathbf{x}+\xi, t+\tau)}\,. \tag{2}$$

The mean square of the magnetic field fluctuations is given by

$$\overline{\mathbf{B}'^2(\mathbf{x}, t)}=B_{ii}(\mathbf{x}, 0, t, 0)\,. \tag{3}$$

As shown in mean-field electrodynamics (Krause and Rädler, 1971; Roberts, 1972; see furthermore the translation of papers of F. Krause, K.-H. Rädler and M. Steenbeck by P. H. Roberts and M. Stix, 1971) $\mathbf{B}'$ can be expressed by an integral

$$\mathbf{B}'=\int \mathbf{G}\mathbf{u}'\bar{\mathbf{B}}\,, \tag{4}$$

if $\bar{\mathbf{B}}$ is not to large. $\mathbf{G}$ represents some Greens tensor, which can be given explicitly in case of the second order correlation approximation (Krause, 1973). Starting from (4) one derives for the magnetic flield correlation tensor of second rank an integral expression of the type

$$B_{ij}=\int \mathbf{G}\mathbf{Q}\bar{\mathbf{B}}\bar{\mathbf{B}}\,. \tag{5}$$

A more explicit evaluation of (5) can be given on some assumptions. If the length scale of $\bar{\mathbf{B}}$ is large compared with the correlation length and furthermore the turbulent velocity field is homogeneous and isotropic one finds

$$\overline{\mathbf{B}'^2(\mathbf{x}, t)}=\frac{1}{3}\iint \left(k^2\hat{Q}_{mm}(\mathbf{k}, \omega)+\right.$$
$$\left.+\hat{Q}_{mr}(\mathbf{k}, \omega)\, k_m k_r\right)\frac{\mathrm{d}\mathbf{k}\,\mathrm{d}\omega}{\eta^2k^4+\omega^2}\,\bar{\mathbf{B}}^2\,. \tag{6}$$

$\hat{Q}_{ij}(\mathbf{k}, \omega)$ denotes the Fourier-transform of $Q_{ij}(\xi\tau)$, and $\eta=1/\mu\sigma$ the magnetic diffusivity.

The self-consistence of the theory, namely that the ratio $\overline{\mathbf{B}'^2}/\bar{\mathbf{B}}^2$ is non-negativ, is guaranteed by the Bochner theorem, which ensures that the quadratic form is positive definit:

$$Q_{ij}(\mathbf{k}, \omega)\, X_i X_j^* \geqslant 0\,; \tag{7}$$

X_i is an arbitrary set of three complex numbers and $X_i^\dagger$ the conjugate one.

We evaluate (6) for a simple model where the longitudinal correlation function $f(\mathbf{x}, \tau)$ is given by

$$f(\xi, \tau) = u^2 \exp\left\{-\frac{\xi^2}{L^2} - \frac{\tau^2}{T^2}\right\}; \qquad (8)$$

L denotes the correlation length and T the correlation time. We find

$$\overline{\mathbf{B}'^2} \approx R_m^2\, \bar{\mathbf{B}}^2 \quad \text{for} \quad \mu\sigma L^2 \ll T\,, \qquad (9)$$

and

$$\overline{\mathbf{B}'^2} \approx R_m\, \bar{\mathbf{B}}^2 \quad \text{for} \quad \mu\sigma L^2 \gg T\,. \qquad (10)$$

R_m denotes the magnetic Reynolds number

$$R_m = \mu\sigma u L\,. \qquad (11)$$

The case (9) is mostly realized in laboratory experiments. The case (10) is realized in nearly all cases of cosmic physics. This relation was already found by Steenbeck and Krause (1969), on the basis of physical arguments, and also by Parker (1973).

For the convection zone of the Sun we have

$$R_m \approx 10^4 - 10^5\,. \qquad (12)$$

Consequently the fluctuating field is much larger than the mean field:

$$\mathbf{B}' \approx 10^2\, \bar{\mathbf{B}}\,. \qquad (13)$$

It seems that this result is rather good confirmed by observations: $\mathbf{B}'$ outside of active regions reaches up to values of about 10^2 Gauss, whereas $\bar{\mathbf{B}}$ is of a few Gauss only.

This result illustrates a certain difficulty. Calculations of dynamo models for the Sun provide for a butterfly diagramm but also for the description of the space-time behavior of the mean poloidal field at the Sun's surface. But according to (13) we can hardly determine the mean poloidal field from the observations, since the fluctuations are to large.

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