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A note on the impulse addition of two colliding spherical objects

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Abstract. In collisions between macroscopic spherical-shape objects, these objects are usually regarded as dimensionless, point-like particles. This assumption causes an overestimate of the orbital velocity of mergers, because real bodies have finite dimensions and a part of their orbital impulse is converted to the rotation of the merger. We give a statistical estimate of this impulse and, thus, the merger's velocity vector, which is a better approximation of reality. The provided formula is simple and, therefore, suitable to be used in a robust simulation of, e.g., the planet-formation process.
Key words: planet formation – celestial mechanics

1. The outline of the problem

The formation of planets has been often simulated considering a merging of large planetesimals. Because of the sufficiently strong gravitational attraction, the collision of an impactor with a target object is inelastic, resulting in a single final body - merger. After the merging, this new object continues in its motion, whereby its velocity vector is derived from the vectorial sum of the original orbital impulse vectors of the precursors.

In a first approximation, the colliding bodies are regarded as dimensionless, point-like particles. Let us consider an inelastic collision of two point-like particles (or a head-on collision of two spherical objects) having the masses m_1 and m_2. Let the result of the collision is the merging of both particles into a single larger object. If the velocity vectors of particles in an inertial coordinate frame, e.g. heliocentric frame, are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then the orbital impulse of the merger in this frame can simply be calculated as

$$\boldsymbol{p}_m = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2. \qquad (1)$$

This can, however, cause an overestimate of the final impulse of the merger. In reality, the sizes of colliding objects are finite and head-on collisions rare. A fraction of the impulse delivered by the impactor to the target body has to obviously be converted into the angular momentum (spin) of the merger. (We neglect a conversion of the orbital energy to heat, which is usually not significant.) Its final orbital momentum is, thus, lower than that given by relation

(1). In the following, we sketch the way of the calculation of an approximate merger's mean final orbital momentum in a situation, when the original target object is impacted by many smaller impactors, whereby the distribution of impact sites onto the adjacent half-sphere of the target object is random. This statistical approach is still not exact. Nevertheless, it is a better approximation of the resultant merger's orbital impulse than a simple vectorial sum (1).

2. The impulse in the target-body-centric coordinates

Let us consider the coordinate frame centered on the target object in the moment of the impact event. The impulse carried by the target object is obviously zero and that of the impactor equals

$$\boldsymbol{p}_i = m_2 \left(\boldsymbol{v}_2 - \boldsymbol{v}_1 \right) . \tag{2}$$

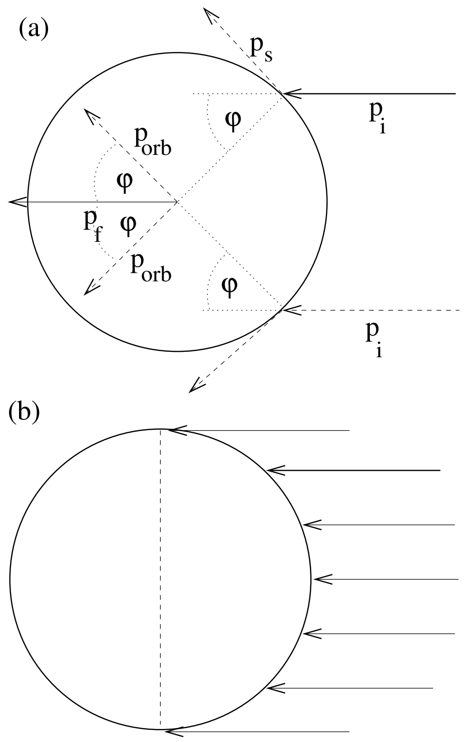

Figure 1. The scheme to the calculation of the impulse of the merger appearing as a result of the collision of finite-sized objects. The gravitational attraction of the target object on the impactor is neglected (see text).

If the impact angle is equal to φ (see Fig. 1a), the impulse delivered by the impactor, p_i, is decomposed into the component p_s that tends to change the spin of the target body and component p_{orb} changing the orbital momentum of the target body, whereby

$$p_{orb} = p_i \cos\varphi. \tag{3}$$

In a simulation, only a tiny fraction of the real collision events is simulated. In reality, a large number of similar impacts, from a given direction, would occur on the half-sphere with the pole in direction given by the unit vector parallel to $-\boldsymbol{p}_i$. We can expect a random distribution of the impact sites on this sphere (see scheme in Fig. 1b).

The distribution of the real impacts is obviously stochastic, with some statistical fluctuations. In various studies, the authors are nevertheless usually interested in a statistically typical collision event. In such a case, it is more reasonable to consider the mean impulse on the given half-sphere than the impulse of the appropriate head-on collision. In the calculation of the mean impulse, there exists a symmetric impact delivering again the impulse p_i with the component p_{orb} of the same size (the dashed arrow labelled as p_i in Fig. 1a). So, the orbital momentum of the target object is enlarged, by the given impactor, about p_f component (Fig. 1a), which is the component of impulse p_{orb} in the direction of impulse p_i. This component must be taken twice, since the same contribution comes from the symmetrical counterpart of p_i. Therefore,

$$p_f = 2p_{orb} \cos\varphi = 2p_i \cos^2\varphi. \tag{4}$$

When we average the impact over the entire half-sphere (Fig. 1b), the size of the total mean impulse, p_A, is

$$p_A = \frac{1}{2\pi R^2} \int_0^{\pi/2} 2p_i \cos^2\varphi 2\pi R^2 \sin\varphi \, d\varphi = \frac{2}{3} p_i = \eta p_i. \tag{5}$$

Notice that we denoted the factor of 2/3 by symbol η. Since the merger acquired the impulse $p_A \neq 0$ in the target-object centered coordinate frame, it moves with velocity

$$\boldsymbol{V} = \frac{p_A}{m_1 + m_2} \tag{6}$$

in this frame. Using relations (5) and (6), we can find

$$\boldsymbol{V} = \frac{\eta m_2}{m_1 + m_2} \left(\boldsymbol{v}_2 - \boldsymbol{v}_1\right). \tag{7}$$

3. The resulting heliocentric impulse

In the heliocentric coordinate frame the merger moves with velocity $\boldsymbol{V} + \boldsymbol{v}_1$ and its impulse is

$$\boldsymbol{P}_m = (m_1 + m_2)\left(\boldsymbol{V} + \boldsymbol{v}_1\right) = m_1 \boldsymbol{v}_1 + (1 - \eta)\, m_2 \boldsymbol{v}_1 + \eta m_2 \boldsymbol{v}_2. \tag{8}$$

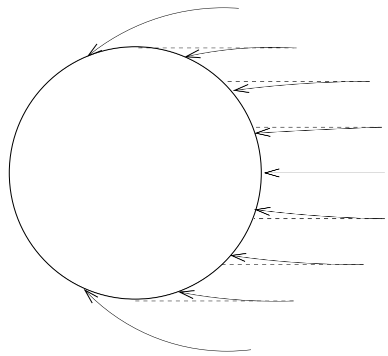

Figure 2. The scheme showing the impulse delivered to the merger by a set of impactors, when the gravitational attraction of the target object on the impactors cannot be neglected (see text).

We note, the result given by relation (8) can be regarded as a good approximation, when the relative velocity, $\boldsymbol{v}_2 - \boldsymbol{v}_1$, between the impacting and target objects is very large, therefore the bending of the impactor trajectory toward the centre of the target object by its gravity can be neglected. If such neglection is impossible, the factor η in (8) differs from 2/3. In Fig. 2, there is shown a scheme of the trajectories of impactors immediately before the impact events (the bending is overestimated in comparison to a common real situation) in such a case. We can see that the direction of a given impact is mostly oriented nearer to the centre of target body that in the case described in Sect. 2. Therefore, the vectorial sum must be different (lower) from the sum which is described in Sect. 2 and illustrated in Fig. 1. In the actual, more general case, the value of factor η depends on the magnitude of relative velocity, $|\boldsymbol{v}_2 - \boldsymbol{v}_1|$, and masses of impactors and the target body.

4. Conclusion

The formula (8) giving the impulse and, at the same time, velocity vector (as $\boldsymbol{V}_{helio} = \boldsymbol{P}_m/(m_1 + m_2)$) of the merger is a better approximation of the real impulse vector than the trivial vectorial sum (1). This formula is not exact either. Nevertheless, the calculation of the exact velocity vector in the given case is a relatively large procedure, therefore, if one wishes or is forced, e.g. in a robust computational simulation, to use a simple relation, the found formula is a good compromise.

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