Contrib. Astron. Obs. Skalnaté Pleso **37**, 83 – 106, (2007)

Radar meteors range distribution model

I. Theory

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Received: January 23, 2007; Accepted: April 30, 2007

Abstract. The theoretical construction of a model of the range distribution of radar meteors is presented. It provides us with possibility of the computation of five important parameters connected with the structure of meteor showers as well as physical features of meteoroids related to them. These are: the mass distribution index, s, the shower flux density, Θ_{m_0}, the ionization coefficient, β, the Levin's parameter (self-similarity parameter), μ, the product of the shape-density coefficient, K, and the ablation parameter, σ. We present details of the numerical model in the article which proved to be a good explanation for the real observed process. The model (RaDiM) is based on the overdense echoes of shower meteors. It makes use of the long-term series of data that have been collected by the Ondřejov meteor radar during almost five decades. Its application and relevant results will be described in subsequent articles.

Key words: physics of meteors – radar meteors – range distribution – physical parameters of meteoroids

1. Introduction

The Ondřejov meteor radar began its operation in 1958 (Plavcová and Šimek, 1960). Since then the valuable long-term series of data have been managed to accumulate. The research has preferably been focused on four meteor showers: Quadrantids, Perseids, Leonids and Geminids. Besides, the other meteor showers have irregularly been observed as well, e. g., Giacobinids during its increased activity in 1998 (Šimek and Pecina, 1999). Because the observations carried out by the Ondřejov meteor radar are only single-station ones scientific research has mainly been concentrated on activity monitoring of selected meteor showers and determinations of their mass distribution indices and fluxes. However, a method was developed in an attempt to connect the Ondřejov observations with physical properties of meteoroids as well. The method is called the Range Distribution Model (hereinafter referred to as the RaDiM). The principle of the RaDiM is based on knowledge of the observed range distributions of overdense radar echoes belonging to meteor showers. In a series of four articles we present details of the model that allows us to determine some physical and chemical properties of the meteoroid as well as the parameters connected with the inner structure of meteor showers. The first paper (I) deals with the RaDiM itself and the process of collection and processing of input data. The other ones are concerned with results. The second (II) is about the shower flux density and the mass distribution index, the third (III) discusses a self-similarity parameter, a shape-density coefficient and an ablation parameter while the last (IV) treats an ionization coefficient and its velocity dependence. We have already published the first simplified form of the model and its applications to two daily showers belonging to the Taurid complex showers, ζ Perseids and β Taurids (Pecinová and Pecina, 2005). However, there was a slight discrepancy between that form of the theory and the observations there as mentioned hereinafter. So, it was necessary to improve and extend the model in order to explain the observed phenomena better. We succeeded in that and we present here an improved completed theory that explains the observed reality very well.

2. Range distribution

Each radar echo registered by the Ondřejov meteor radar is characterized by four quantities. These are: the time instant of the echo occurrence with an accuracy of second, the time behaviour of the echo amplitude, the echo duration (up to several minutes) and the range of a reflecting point on a meteor trail from the radar. The unambiguous measurement of ranges runs into the interval $<60, 300> \cup <360, 600>$ km. The blocking gap between 300 and 360 km on a record arises from an artificial enlargement of measurement of ranges from natural 300 km (corresponding to repetition frequency of 500 Hz) up to 600 km and was installed to suppress recording of the ground based reflections (up to 60 km and periodically each 300 km).

When we sort out observed radar echoes into chosen range intervals according to other characteristics (i.e., an observed time interval, a selected interval of a duration), we can get a column chart similar to the diagram shown in Fig. 1. Thus, we define the range distribution as the dependence of the echo rates on ranges from the radar (the observation site).

The observed range distribution curve associated with a particular meteor shower depends not only on the radar equipment used but also on the radiant position that mirrors the fact that ionized meteor trails associated with a particular meteor shower occur inside a restricted height interval. This interval depends evidently on velocities and masses of meteoroids and also on their other physical properties, e. g., on the ablation parameter σ and the shape-density coefficient K (e. g., Ceplecha et al., 1998). Since during the observations of meteor showers we register simultaneously a lot of meteors with various masses, their mass distribution described by the mass distribution index s together with the shower flux density Θ_{m_0} contribute to the shape of their range distribution curve. It follows that the RaDiM needs to be linked to the quantities that are

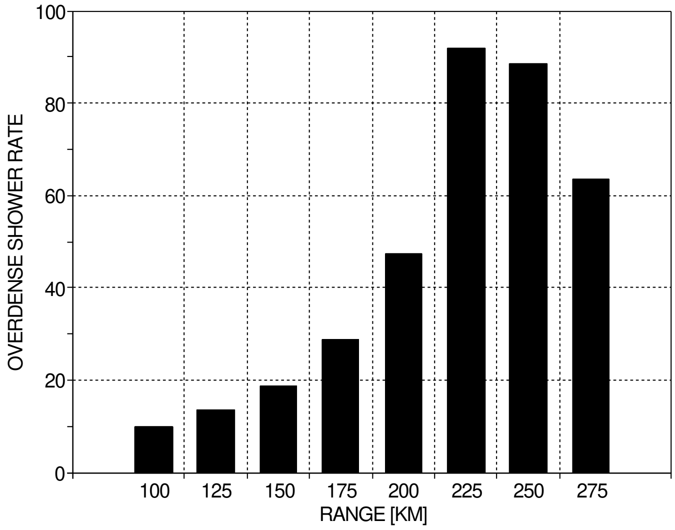

Figure 1. This example of the typical range distribution was constructed from radar meteors recorded during observations of the Geminids, between 23 and 3 UT, on the 13th/14th of December, in 2000. The histogram comprises the overdense echoes with durations exceeding 0.4 s. The vertical axis shows shower rates in particular 25-km-wide range intervals, which are represented by their initial points on the horizontal axis.

associated with the inner structure of the meteor showers and the quantities related to physical characteristics of the investigated meteoroids as well.

3. Range distribution model

3.1. Fundamental formula

As a consequence of the fact that the range distribution is a result of the contribution of shower meteors having various masses, the theoretical model has to be based on the generalization of the well-known mass distribution power law (e. g., McKinley, 1961)

$$d\overline{N} = c\, m^{-s}\, dm, \tag{1}$$

giving the number of meteors having masses within the mass interval $(m, m + dm)$. Here s stands for the mass distribution index, which we suppose to be constant within the whole mass interval we consider and c is a normalizing factor. The law was derived from observations over a large part of the sky. Assuming that it is valid for any element of the echo plane and also for any sufficiently short time interval, we can generalize it in the following way. Apparently, the larger collecting area and the longer time interval the greater number of meteors we should observe. This results in a more general mass distribution power law in the form (Belkovich, 1971):

$$d^3N = c_n\, m^{-s}\, dm\, dS\, dt. \tag{2}$$

Here dt is the element of time interval, $dS = RdRd\vartheta$ is the element of the collecting area within the echo plane expressed in the polar coordinates (R is the distance from the observational site (radar), ϑ is the angle measured within the collecting area of the echo plane, see later). The echo plane is defined as a plane perpendicular to the radiant direction that runs through the observational site. It means that the echo plane is the set of points at which a specular reflection from meteor trails can occur. The collecting area is defined as an intersection of the echo plane with the antenna pattern of a meteor radar. Since a shower radiant moves with time the position of the echo plane changes and consequently the size of the collecting area alters. Received power from all its points has to exceed the minimal power that the radar is able to recognize as a signal from a meteor.

To specify the normalizing factor c_n in (2), we employ the definition of the shower flux density Θ_{m_0}. This quantity expresses the number of meteors crossing a unit surface of the echo plane per time unit having masses in excess of m_o. Mass m_o is an optional constant that will be discussed in detail in Subsection 3.6. The definition together with the law (2) leads to the following relation connecting the shower flux density Θ_{m_0} with the normalizing factor c_n:

$$\Theta_{m_0} = \int\limits_{m_0}^{+\infty} \frac{d^3N}{dS\,dt}\, dm \;=\; c_n \int\limits_{m_0}^{+\infty} m^{-s}\, dm \;=\; \frac{c_n}{s-1}\, m_0^{1-s}. \tag{3}$$

Elimination c_n between equations (2) and (3) yields the important generalized mass distribution power law

$$d^3N = (s-1)\,\Theta_{m_0}\, m_0^{s-1}\, m^{-s}\, dm\, dS\, dt. \tag{4}$$

Obviously, $m \geq m_0$ has to be valid. Since the mass of a meteoroid decreases during its passage through the Earth's atmosphere and the rate of mass loss is different for meteoroids of various sizes, shapes and chemical composition, the mass m in (4) should represent m_∞ of the meteoroid, i. e., its mass before entering the Earth's atmosphere. So, we rewrite (4) as

$$d^3N = (s-1)\,\Theta_{m_0}\, m_0^{s-1}\, m_\infty^{-s}\, dm_\infty\, dS\, dt. \tag{5}$$

In the above equation d^3N stands for an incremental rate with respect to mass. Since it is better to deal with cumulative rates due to greater numbers in practice, our theoretical range distribution model is based on the corresponding cumulative quantity. To answer this purpose, we carry out the integration in (5) with respect to mass from a certain value of m_∞ to $+\infty$. We obtain

$$d^2 N_c = \Theta_{m_0} \, (m_0/m_\infty)^{s-1} \, dS \, dt. \tag{6}$$

Here $d^2 N_c$ is the cumulative number of meteors having masses m_∞ in excess of m_o registered during the time element dt and within the element of the echo plane dS. Equation (6) is the principal formula in the differential form. We get its integral form when integrating with respect to time t and collecting area S_{col}:

$$N_c = \Theta_{m_0} \int_{t_1}^{t_2} dt \int_{S_{\mathrm{col}}} (m_o/m_\infty)^{(s-1)} dS = \Theta_{m_0} \int_{t_1}^{t_2} dt \int_{R_1}^{R_2} dR R \int_{\vartheta_1(R)}^{\vartheta_2(R)} \left(m_o^{1/3} / m_\infty^{1/3} \right)^{3(s-1)} d\vartheta. \tag{7}$$

This is the fundamental formula of the RaDiM in the integral form. Since it is not possible to observe the quantity m_∞ in a direct way we have to transform (7) to include only a suitable measurable quantity. This will be carried out in the next subsection.

3.2. The used radar quantity

Now we need to pay attention to the replacement of m_∞ in the fundamental formula (7) by a directly observable quantity with a clear relation to the mass of a meteoroid. Generally, there are two ways how to make use of the data we get from the observations. We can choose either the duration of an observed radar echo or its amplitude. Both quantities are generally connected with the electron line density α_e at the specular point on a meteor trail and, therefore, with mass m_∞ (e. g. McKinley, 1961). Unfortunately, we are not able to make use of the amplitudes of registered echoes due to the fact that the observations are only single-station ones without an interferometer. As a consequence, it is impossible to determine the position of the registered echo within the antenna pattern and make the relevant correction of its amplitude. The second possibility is to work with durations. They are handy because their observed values are not influenced by their positions within the antenna pattern. Now we need to make a choice between the duration of underdense and overdense echoes.

The decay of the electron volume number density that has a strong influence upon an echo duration can be a result mainly of three effects: ambipolar diffusion, recombination and attachment of free electrons to neutral particles. We take into account only ambipolar diffusion. The reason for the neglect of the remaining two effects is examined in Subsection 3.3. Let us now consider only ambipolar diffusion. In this case the duration T_U of the underdense echo that is defined as the time constant of the exponential drop of the amplitude of received signal is given by (e. g. Bronshten, 1983):

$$T_U = \frac{\lambda^2}{16 \, \pi^2 \, D(h)}. \tag{8}$$

Here λ is the wavelength of the electromagnetic wave the radar transmits (8 m) and $D(h)$ stands for the ambipolar diffusion coefficient which is height dependent. Equation (8) expresses one important fact: T_U is not related to any physical characteristics of meteors at all. Thus, the only usable observed quantity for our purpose is duration T_D of the overdense echo. The connection between T_D, the electron line density (the number of electrons created per unit path length), α_e, and the initial radius of a meteor trail at the reflecting point at the height h (in the case of the ambipolar diffusion process) is given by (e. g., Bronshten, 1983):

$$T_D = \left(\frac{\lambda}{2\pi}\right)^2 r_e \frac{\alpha_e(h)}{D(h)} - \frac{r_o^2}{4D(h)}, \tag{9}$$

where r_e is the classical radius of electron ($r_e = 2.81 \times 10^{-15}$m), and r_o denotes the initial radius of a meteor trail. It is obvious from (9) that the quantity T_D is very suitable for our purpose. We derive the desired connection between T_D and m_∞ in Subsection 3.5 by means of (9).

The bottom limit of the duration we take is 0.4 s. This option follows from equation (8). The slowest shower we deal with is the Giacobinid one. Its preatmospheric speed is about 23 km s^{-1} (Lovell, 1954) and the corresponding characteristic height of its radar meteors is 87 km (McKinley, 1961). The diffusion coefficient at this height (computed according to CIRA (1972) for October and relation (15)) has the value of 1.57 m^2 s^{-1}. Thus, the underdense duration computed from (8) is approximately 0.25 s. The remaining examined showers are faster and consequently their characteristic radar heights are greater. Since the value of the ambipolar diffusion coefficient increases with increasing height, the value of 0.25 s is the maximum of (8). To be on the safe side and avoid the transient type of radar echoes the bottom limit of the duration of the overdense echoes we accept is 0.4 s.

Since we use cumulative rates the upper limit of echo duration should be infinity according to (7). However, in practice there is some maximum value in the observed data. This value varies from one examined shower to another and usually does not exceed approximately one minute. This fact also enables us to avoid dealing with the effect of attachment. For example, no activity of ζ Perseids and β Taurids in the echo duration category exceeding 10 s was observed, the corresponding limit for autumn Taurids was 5 s (Pecina et al., 2005). There are some exceptions, e. g., Leonids (Pecina and Pecinová, 2004) when we have recorded echoes with durations of order of minutes. But it is an exceptional case and a contribution to a whole cumulative rates are almost on the zero level for those echoes. To sum up, the upper limit of the duration we work with is determined for each individual meteor shower and year of its occurrence.

Furthermore, it should be emphasized that the RaDiM is able to work with arbitrary limits of duration (mass). On the other hand, the greater interval of duration the greater number of observed echoes and, consequently, the corresponding range distribution curve is defined better. This is our main reason for using the cumulative rates.

3.3. Dissipation of meteor trails

Electrons recombine with positive ions to form neutral molecules or atoms. This effect can be expected to contribute to an eventual dissipation of the meteor trail. According to Bronshten (1983), the most effective process of recombination, the dissociative one, has the electron recombination coefficient $a_e \simeq (2 \div 4.5) \times 10^{-13}$ $\mathrm{m}^3\ \mathrm{s}^{-1}$. As a consequence of this small value, recombination can hardly be a significant factor in comparison with ambipolar diffusion. Hence, in order that recombination could play significant role within the observed meteor trains, a_e should be greater more than 10^4 times in comparison with the previous value.

Some electrons may attach themselves to neutral molecules to create negative ions. This process depends on the constant b_e of attachment and on the concentration n of molecules to which electrons attach. In the past molecular oxygen was suspected as one of the most probable molecules involved in the creation of negative ions (McKinley, 1961). The coefficient of attachment is not well determined but one can find in literature (Bronshten, 1983) that its value is small at meteoric heights around 100 km (the heights in question). Thus, the attachment process to oxygen does not have noticeable influence upon the decay of number density of electrons. It follows that attachment may become relevant at heights bellow 75 km. On the other hand, Bibarsov (1972) proposed that the attachment of electrons to neutral particles of meteor origin, i.e., particles ablated from a meteoroid's surface, happens rather than to oxygen. His view was not accepted by the scientific community. Recently, ozone has been taken into account (e. g., Jones et al., 1990). However, because of $b_e \simeq 10^{-18}\ \mathrm{m}^3\,\mathrm{s}^{-1}$ in this case (Baggaley, 1972) and concentration of ozone molecules $n \simeq 10^{15}\ \mathrm{m}^{-3}$ (at its maximum at 85 km), even ozone cannot play a significant role in our observations. This can easily be seen from the following relation connecting the duration T_D influenced only by ambipolar diffusion and the observed duration T that can be affected by both processes. The function $T_D(T)$ has the form (e. g., Bronshten, 1983):

$$T_D = \left(T + \frac{r_0^2}{4D} \right) \exp(b_e\, n\, T) - \frac{r_0^2}{4D}. \tag{10}$$

The term $\exp(b_e\, n\, T) \simeq 1$ for ozone and we can simplify this to $T_D \simeq T$ and use directly observed duration T in (9) instead of T_D.

To conclude, both effects can be expected to contribute to the eventual dissipation of the meteor ionization within the examined trails but the rates at which they operate revealed that they are not significant in comparison with ambipolar diffusion. Moreover, both phenomena should occur at rather lower heights while our echoes originate at greater ones. Thus we consider only ambipolar diffusion in the RaDiM.

3.4. Assumptions

We infer the explicit functional dependence of m_∞ on the duration T_D from the physical theory of meteors under several assumptions.

(i) We neglect deceleration of meteoroids. The fact that deceleration of radar echoes of the type the RaDiM uses can be neglected is based on literature (e. g., Kashcheev et al., 1967; Voloshchuk et al., 1989).

(ii) Meteoroids behave in a more complicated way during their passage through the Earth's atmosphere than the classical single-body theory predicts. This proved to be true in many cases in the past (e. g., Campbell-Brown and Koschny, 2004) and also in our investigation. At first, we tried to develop a simple model of the range distribution when taking into account the assumption that a meteoroid, originally of a spherical shape, does not change its shape during its passage through the Earth's atmosphere. That model of first approximation was applied to the data of two daytime showers of the Taurid complex (Pecinová and Pecina, 2005). In that case it worked fairly well but there was still some discrepancy between the observed and theoretically gained range distributions. So, we were obliged to improve our model. In accordance with Levin (1956) we consider the law governing a variability of the cross-section of a meteoroid during ablation. We define this law in terms of Levin's parameter μ (also self-similarity parameter)

$$S = S_\infty \, (m/m_\infty)^\mu \, . \tag{11}$$

Here, S is the instantaneous cross-section of a meteoroid and m its corresponding mass. Symbol ∞ denotes its preatmospheric value. Obviously, if the ablating body remains self-similar (the classical single-body theory), $\mu = 2/3$ is valid. Apparently, the physical meaning has only the affirmative values of μ so that we take it from the interval $<0, +\infty)$. The usage of μ improves the fitting of theoretical range distribution to the observed one as is treated in more detail in Paper III. Further, we consider the preatmospheric shape of a meteoroid to be a sphere. Thus, we connect S_∞ with m_∞ and the bulk density of a meteoroid δ by the relation

$$S_\infty = A \, (m_\infty/\delta)^{2/3} \, . \tag{12}$$

The symbol A stands for a numerical constant. In this case: $A = \pi(3/(4\pi))^{2/3} \doteq 1.21$.

(iii) We use an exponential dependence of the air density ϱ on the height h in the widely-used form (e. g., Pecina and Ceplecha, 1983)

$$\varrho(h) = \varrho_o \exp{(-h/H)}. \tag{13}$$

The constants ρ_o and H result from the least-square fit of this dependence to the atmosphere CIRA (1972) within the meteor heights interval 80-120 km. It is performed separately for each relevant month in which the examined meteor shower occurs.

(iv) The term (9) includes the initial radius r_o. We use the following model of r_o (Baggaley, 1970):

$$r_o = r_{oo}\,(\rho_k/\rho(h))^f\,(v(h)/v_k)^g\,. \qquad (14)$$

The constants entering (14) are:

- $r_{oo} = 1.5$ m,
- $\rho_k = 0.5306 \cdot 10^{-6}$ kg m^{-3} (at 100 km),
- $v_k = 40$ km s^{-1},
- $f = 0.45$,
- $g = 0.57$.

The model (14) points out that the initial radius depends on height through the density of air and the velocity of meteoroids.

The term $r_o^2/4D(h)$ in (9) depends also on the ambipolar diffusion coefficient D. In further consideration we make use of the relation

$$D\,\varrho = D_r\,\varrho_r, \qquad (15)$$

where D_r and ϱ_r mean the values at a reference height. We accept the value $D_r = 4.2$ m^2 s^{-1} valid for height of 93 km in our computations (Belkovich, 1971). By substituting (15) into (14) we gradually get:

$$\frac{r_o^2}{4D(h)} = \frac{r_{oo}^2}{4D_r\rho_r}\,\rho(h)\left(\frac{\rho_k}{\rho}\right)^{2f}\left(\frac{v}{v_k}\right)^{2g} = c_1\,\rho^{1-2f}\,v^{2g}, \qquad (16)$$

where $c_1 = \left[\frac{r_{oo}\,\varrho_k^f}{2\,v_k^g}\right]^2 \frac{1}{D_r\,\varrho_r}$ remains constant. Consequently (9) takes the form

$$T_D = \left(\frac{\lambda}{2\pi}\right)^2 r_e\,\frac{\alpha_e\,\varrho}{D_r\varrho_r} - c_1\,\rho^{1-2f}\,v_\infty^{2g}. \qquad (17)$$

Since $T_D \gg c_1\,\rho(h)^{1-2f}\,v_\infty^{2g}$ holds true for Baggaley's values of constants this term does not influence the computed values noticeably. In spite of this we do not neglect it in our computations to be able to take it into consideration when some other r_o-model is accepted.

3.5. Connection between m_∞ and T_D

The aim of this subsection is to derive the quantity m_∞ as a function of the observed duration T_D of the overdense echo under the above mentioned assumptions. For this purpose we need to find the expression of α_e as a function of m_∞ and then substitute for it into (9). This is performed in three steps.

Firstly, we proceed from the mass-loss equation (or the ablation one) describing the process of ablation. Ablation is defined as any removal of meteoroid mass via its passage through the Earth's atmosphere in the form of gas, droplets or solid fragments. It takes the form (e. g., Bronshten, 1983)

$$\frac{dm}{dt} = -\frac{\Lambda}{2Q} S \varrho v^3. \tag{18}$$

Here Λ is called a heat-transfer coefficient. Since the energy used on ablation cannot evidently exceed the total kinetic energy of the oncoming stream of molecules, this dimensionless coefficient is less than or equal to unity. Apart from the energy used for heating and ablation of the meteoroid mass dm, some part of the energy of the impinging molecules is consumed by heating up of the meteoroid itself, another part is converted into radiation and ionization of atoms and molecules of both the meteoroid and the air and also a significant portion of energy is dissipated by reflected air molecules and vapor molecules and atoms. Introducing a very important parameter σ, known as the ablation parameter, by the relation

$$\sigma = \frac{\Lambda}{2Q\Gamma} \qquad \left([\sigma] = \mathrm{s}^2\,\mathrm{km}^{-2}\right), \tag{19}$$

we can rewrite equation (18) into the form

$$\frac{dm}{dt} = -\sigma \Gamma S \varrho v^3. \tag{20}$$

The coefficient Γ is a drag coefficient. Because we take the atmospheric profile in the form of (13), this also implies

$$dh = -(H/\varrho)\, d\varrho. \tag{21}$$

To proceed further, we use the following relationship defining a geometric dependence of the height of the meteoroid flight h on time t (e. g., Bronshten, 1983):

$$\frac{dh}{dt} = -v \cos z_R, \tag{22}$$

where z_R is the zenith distance of a shower radiant. We usually suppose that $\cos z_R$ is independent of height, h, because the variability of z_R can be proved to be significant only in case of a very long bright meteor the trajectory of which extends over a large part of the Earth's surface. This is not the case of radar echoes registered by means of the Ondřejov meteor radar. Moreover, this relation does not take into account the curvature of the Earth's surface, which is also insignificant from the point of view of the data we use. The length of the meteoroid flight l connects with time t by the relation $dl = v\,dt$. Combining equations (21) and (22) we get the important relation:

$$dt \;=\; (H/\varrho\, v \cos z_R)\, d\varrho. \tag{23}$$

At this point there is necessary to modify the mass-loss equation due to Levin's proposition. Let us now substitute relations (11) and (23) into ablation equation (20). After making necessary adjustments with regard to (12) and the non-deceleration assumption we arrive at the expression

$$\left(\frac{m}{m_\infty}\right)^{-\mu} d\left(\frac{m}{m_\infty}\right) \;=\; -\,\frac{HK\sigma v_\infty^2}{m_\infty^{1/3}\cos z_R}\, d\varrho. \tag{24}$$

Here we make use of the definition of the shape-density coefficient $K = A\,\Gamma/\delta^{2/3}$. Subsequent integration in (24) provides us with

$$m \;=\; m_\infty \left\{1-(1-\mu)\,\frac{HK\sigma v_\infty^2}{m_\infty^{1/3}\cos z_R}\,\varrho\right\}^{\frac{1}{1-\mu}}. \tag{25}$$

When we substitute this result into (20), we obtain

$$\frac{dm}{dt} \;=\; -K\sigma m_\infty^{2/3} v_\infty^3\, \varrho \left\{1-(1-\mu)\,\frac{HK\sigma v_\infty^2}{m_\infty^{1/3}\cos z_R}\,\varrho\right\}^{\frac{\mu}{1-\mu}}. \tag{26}$$

Secondly, we turn out our attention to meteor ionization. The fact that meteoroids during their passage through the Earth's atmosphere leave an ionized conducting path provides us with possibility of studying them by means of the radar. The formation of an ion-electron trail is a consequence of inelastic collisions between the evaporating atoms of a meteoroid and air molecules and atoms. The trail is supposed to be quasi-neutral as a whole. One of the most important feature of the trail we work with is the electron line density α_e that appears in the ionization equation (e. g., Bronshten, 1983)

$$\alpha_e = -\frac{\beta}{\mu_a\, v}\left(\frac{dm}{dt}\right), \tag{27}$$

where the symbol μ_a stands for the average mass of a meteoroid atom. We adopt the value $\mu_a = 40\times\mu_H$ ($\mu_H = 0.16735\times10^{-26}$ kg is the mass of hydrogen atom) after Ceplecha et al. (1998). The symbol β is called the ionization coefficient or the ionization probability (dimensionless quantity) and equals to an average number of free electrons released during collisions of one evaporated meteor atom with other particles. The notation of the equation expresses the fact that ionization comes from meteor atoms but not from the atmospheric particles. The quantity β depends on meteor velocity in a way that has not been known definitely yet. We get the desired dependence of the electron line density α_e on height h, the ionization curve, by substituting dm/dt from the modified ablation equation (26) into (27) with regard to the first assumption. This yields

$$\alpha_e = \frac{K\sigma m_\infty^{2/3} v_\infty^2}{\mu_a} \beta\, \varrho(h) \left\{ 1-(1-\mu)\, \frac{HK\sigma v_\infty^2}{m_\infty^{1/3} \cos z_R}\, \varrho(h) \right\}^{\frac{\mu}{1-\mu}}. \tag{28}$$

This relation expresses a well-known fact that α_e depends not only on height h via the air density ϱ but also on parameters σ, K, β, μ and the initial values m_∞, v_∞.

Thirdly, we substitute from (28) into (17) to obtain the desired relationship connecting observed T_D with physical quantities of meteoroids:

$$(T_D + c_1 \rho(h)^{1-2f} v_\infty^{2g}\,)\; x^2 \; = c\, a\, \varrho(h)^2 \left\{ 1 - (1-\mu) \frac{b}{\cos z_R(t)}\, \varrho(h)\, x \right\}^{\frac{\mu}{1-\mu}} \tag{29}$$

In this transcendental equation for x the symbols x, a, b, c designate:

$$x \;=\; m_\infty^{-1/3}, \tag{30}$$

$$a \;=\; K\,\sigma\, v_\infty^2\, \beta(v_\infty), \tag{31}$$

$$b \;=\; K\,\sigma\, v_\infty^2\, H, \tag{32}$$

$$c \;=\; \left(\frac{\lambda}{2\pi}\right)^2 \frac{r_e}{\mu_a\, D_r\, \rho_r} \;=\; const. \tag{33}$$

The constant c depends on the used equipment only via λ and z_R is the zenith distance of a shower radiant. Furthermore, it is important to note that the parameters a and b depend namely on the physical properties of meteoroids, that is, on the shape-density coefficient K, the ablation parameter σ, and on the ionization coefficient β. We suppose both quantities a and b to be the same for all members of a particular meteor shower. It means that all members belonging to the same meteor shower have the same physical properties. By means of (31), (32) and (33) we introduce the following useful relations:

$$K \cdot \sigma \;=\; \frac{b}{H\, v_\infty^2}, \tag{34}$$

$$\beta(v_\infty) \;=\; H\, \frac{a}{b}. \tag{35}$$

Unlike quantities a and b, these relations have the evident physical meaning.

3.6. Optional constant m_o

The quantity m_o that is included in (7) does not depend on the position of a trail reflecting point within the collecting area and is, therefore, constant with respect to the integration. We look for the minimum mass related to the bottom limit of the overdense duration, 0.4 s, and accept its value of 10^{-5} kg. It is estimated in the following way.

The highest electron density occurs at the maximum of the ionization curve. Obviously, the higher the mass of a meteoroid the higher the electron line density and vice versa. Moreover, the reflection at other point than at that of maximum ionization requires a higher value of mass to yield the same signal strength. So, in order to estimate the value of the minimum mass m_o we need to express a relation between the maximum electron line density and the corresponding mass. Therefore, we calculate the derivative of (28) with respect to ϱ and put the result zero. The result reads

$$\varrho_{\max} = \frac{m_\infty^{1/3} \cos z_R}{H\, K\, \sigma\, v_\infty^2}. \tag{36}$$

The height $h_{\max}$ at the point of the maximum ionization is

$$h_{\max} = H \ln \left(\frac{m_\infty^{1/3} \cos z_R}{H\, K\, \sigma\, v_\infty^2} \right). \tag{37}$$

It is interesting to note that the $\varrho_{\max}$ and consequently $h_{\max}$ do not depend on the self-similarity coefficient μ. The electron line density $\alpha_{\max}$ at the point in question is

$$\alpha_{\max} = \frac{\beta(v_\infty)\, m_\infty \cos z_R}{H\, \mu_a}\, \mu^{\frac{\mu}{1-\mu}}. \tag{38}$$

Due to the fact that the electron line density of overdense trail relates to the duration, T_D, via (9) we get:

$$m_\infty = \frac{H\, \mu_a}{\beta(v_\infty) \cos z_R} \left(\frac{2\pi}{\lambda} \right)^2 T_D\, \frac{D(h)}{r_e}\, \mu^{\frac{\mu}{1-\mu}}. \tag{39}$$

The term $r_o^2/4D$ was neglected for the reason described in Subsection 3.4. Now we can estimate m_o. We look for the minimum mass that relates to the duration of 0.4 s. As we explicated in Subsection 3.3, this duration corresponds to the height level of 87 km and $D(87\text{km}) = 1.57\ \text{m}^2\ \text{s}^{-1}$. For our estimate we need to accept some model of the ionization coefficient β. We take a semiempirical model of $\beta(v_\infty)$ by Kashcheev et al. (1967) (being probably most frequently in use) that serves us also for the initial estimation of this quantity during our computations:

$$\beta = \beta_k v^n, \tag{40}$$

The constants are: $\beta_k = 0.12649 \times 10^{-6}$ and $n = 3.5$ ($[v]$ =km s^{-1}). The ionization coefficient which corresponds to the Giacobinids: $\beta = 0.00738$. Further, the minimum value of (39) occurs when simultaneously the quantity $\mu^{\frac{\mu}{1-\mu}}$ takes the minimum value and the inverse of $\cos z_R$ takes the maximum value. The function $f(\mu) = \mu^{\frac{\mu}{1-\mu}}$ is monotonously increasing within the interval $<0, +\infty)$ so that its minimum: $f(0) = 1$. The maximum value of $\cos z_R$ equals 1. The values of the other constants are: $H = 5.409$ km (computed in the case of Geminids by means of CIRA (1972), $\lambda = 8$ m, $r_e = 2.81 \times 10^{-15}$ m, $\mu_a = 40 \times 0.16735 \times 10^{-26}$ kg. Hence, all of this yields: $m_o \doteq 0.66 \times 10^{-5}$ kg. This value holds true at the point of maximum ionization. At other points which do not coincide with the maximum one mass must be even higher. Since the reflection exactly at the point of maximum ionization is rather exceptional and because the possible values of $f(\mu)$ in practice are greater than 1 we accept the limiting mass to be $m_o \simeq 1 \times 10^{-5}$ kg without introducing any substantial error into our considerations.

4. Computation

Equation (7) expresses the fact that we observe meteors crossing the collecting area of the echo plane $S_{\rm col}$ during the time interval (t_2, t_2). In other words, we integrate over the time interval during which the observation was carried out and over the domain of the echo plane from which the radar is able to register radar echoes. The limits of the integration are the following:

1. t_1, t_2 are time limits of observational interval (optional),
2. R_1, R_2 are limits of particular range interval (optional),
3. ϑ_1, ϑ_2 are bounds of angular interval within the collecting area $S_{\rm col}$ depending on the range from radar and on the radiant position.

Thus, after choosing the time and the range limits we need to determine the angular limits. The angles can be computed with the assistance of the radar equation valid for overdense echoes (e. g., Kaiser, 1961):

$$P_R = P_T \, \frac{\lambda^3}{54\pi^3 R^3} \sqrt{r_e} \, G^2(\vartheta) \, \sqrt{\alpha_e(h)}. \tag{41}$$

The radar equation connects the received power P_R and transmitted one P_T, (under the assumption we have the common antenna for transmission and receiving) after the reflection on the overdense trails. $G(\vartheta)$ is the antenna gain and R is the range of the specular point on the trail from the radar. The condition for the computation of the angular limits follows from the fact that received power P_R has to be greater than or equal to limiting power $P_{\rm min}$. This fact together with equation (17) leads to

$$\frac{P_T}{P_{min}} \, \frac{\lambda^2 \, G^2(\vartheta)}{27\pi^2} \, \frac{\sqrt{D_r \rho_r}}{\sqrt{\varrho} \, R^3} \sqrt{T_D + c_1 \, v_\infty^{2g} \, \varrho(h)^{1-2f}} \geqq 1. \tag{42}$$

In the case of the Ondřejov meteor radar $P_{\min} = 2 \times 10^{-13}$ W. We take the functional dependence of the gain G on the angle ϑ inside the collecting area of the echo plane from the antenna pattern by means of the transformation relations. For more details, see Pecina (1982). The relation between height h, ϑ and R is given by the cosine theorem of plane trigonometry

$$h = \sqrt{R_E^2 + R^2 + 2\, R_E\, R\, \sin z_R\, \cos\vartheta} - R_E. \qquad (43)$$

The symbol R_E stands for the Earth's radius. The boundaries ϑ_1, ϑ_2 are limiting values of the region inside which $P_R \geq P_{\min}$. Such limits surely exist since $G(\vartheta)$ is maximum for ϑ_m corresponding to the direction of the antenna pattern vertical plane. When even at $\vartheta = \vartheta_m$ (42) is not valid no signal can be detected. On the other hand $G(\pm\pi/2) = 0$ because these limits correspond to the observational line coinciding with local horizon. Since the dependence of the left hand side of (42) on ϑ is monotonous within both the interval $\vartheta \in <-\pi/2, \vartheta_m>$ and $\vartheta \in < \vartheta_m, +\pi/2>$ with the maximum reached just at $\vartheta = \vartheta_m$, the existence of both angular limits is justified.

The numerical procedure applied to perform the integration in (7) is the following. The time interval $< t_1, t_2 >$ was divided into a set of subintervals long 1 hour at the most. The results got on all subintervals were added together to receive the final result. The integration within each subinterval was performed using the Gauss quadrature formula (e. g., Kopal, 1955) integrating the polynomial of the 7th order exactly. The integration with respect to range was carried out separately on each interval $<R_1, R_2>$ and all results were summed together to provide the result. On each subinterval $<R_1, R_2>$ the Gauss formula integrating the polynomial of 3rd order exactly was employed. The integration with respect to ϑ within the interval $<\vartheta_1, \vartheta_2>$ was performed by dividing this interval into a number of subintervals being 1° long and then adding all partial results to get the final result. The integration within each subinterval was done by means of the Gauss formula for the polynomial of 3rd order. We get the parameters $\Theta_{m_0}, s, a\,(K{\cdot}\sigma, \beta), b\,(K{\cdot}\sigma)$ and μ from the least-square fit of the theoretical rates computed according to our principal formula (7) to an observed range distribution as described in Appendix A.

The iterational method of computation of the parameters is the following. Primarily, we define normalized rates of observed rates within a particular range group as the observed rate divided by the rate at the maximum of the observed range distribution. Then these rates do not depend on Θ_{m_0} as it can easily be recognized from (7). They are functions of s, $K{\cdot}\sigma$, μ, β. Numerical computations revealed that the normalized rates depend mainly on s and $K \cdot \sigma$. They depend in a somewhat weaker way on μ and β. Therefore, the process of computation is divided into several substeps. During the first one only s and $K{\cdot}\sigma$ are computed while the remaining ones are kept constant. The starting value of the mass distribution index s we take to be around the value computed from the classical $\log N$ vs. $\log T_D$ fit. The starting value of $K{\cdot}\sigma$ is 0.01 corresponding to $K = 1$

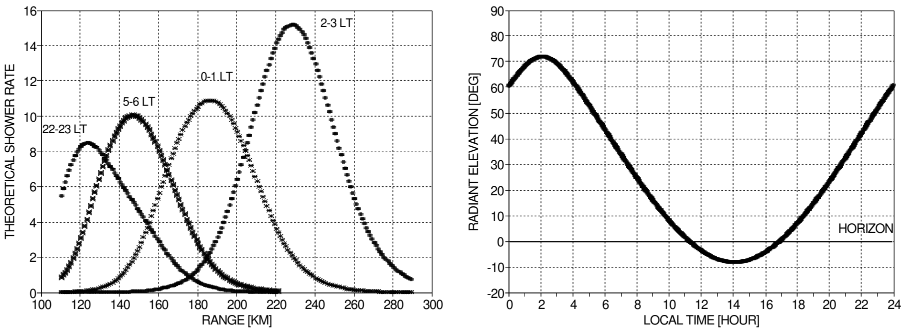

Figure 2. The left hand side picture shows the theoretical range distribution computed for Geminids, as a function of time. All four curves are marked with relevant limits of time interval t_1 and t_2 in LT and were computed for mass $m_o = 10^{-5}$ kg, $v_\infty = 36$ km s^{-1}, $K{\cdot}\sigma = 0.01$ s^2 km^{-2}, $s = 1.5$, $\mu = 2/3$, $\beta = 0.100$, $D_r = 4.2\,\mathrm{m}^2\,\mathrm{s}$ (height of 93 km), $H = 5.409$ km and $\varrho_o = 56.803$ kg m^{-3}. We present on the right hand side picture just for comparison the time course of the elevation of the Geminid radiant.

and $\sigma = 0.01$. The starting value of μ is set to 2/3 and the corresponding value of β is set to the value of this quantity following from the formula of Kashcheev et al. (1967). After getting the s and $K{\cdot}\sigma$ values the next parameter from the above mentioned set is added and so on. Eventually, all these parameters are evaluated. In the second (and last) stage the Θ_{m_0} is calculated where the original (not normalized) rates are used.

The way how a theoretical rate of meteors depends on quantities in question will be visualized in one of the relevant articles dealing with the particular parameter. In this work we focus only on the time and velocity dependence. The time dependence of the theoretical range distribution on time is realized via $\cos z_R$. To visualize this dependence, a few theoretical range distributions valid for 1 hour time intervals were computed for various radiant positions. These distributions in the case of the Geminid radiant are depicted in Fig. 2. The Geminid radiant culminates around 2.5h LT. Obviously, the higher the radiant elevation the greater rates of echoes. Further, the maximum of the range distribution moves to the more distant ranges as the radiant elevation increases. The velocity dependence of RaDiM is not so clearly pronounced because a lot of parameters that enter it are dependent on velocity (e. g. the ionization parameter β, the initial radius r_o). Fig. 3 demonstrates the changes of the range distribution course as parameter v changes. Obviously, in the case of a higher velocity the maximum of the relevant range distribution lies in more distant ranges than in the case of slower meteor showers. This fact corresponds with the course of the ionization curve as it is shown in Fig. 4. This shows that the smaller velocity is the deeper in the Earth's atmosphere the ionization curves begin and cease. Also, the greater value of v the greater value of $\alpha_{\max}$ at the

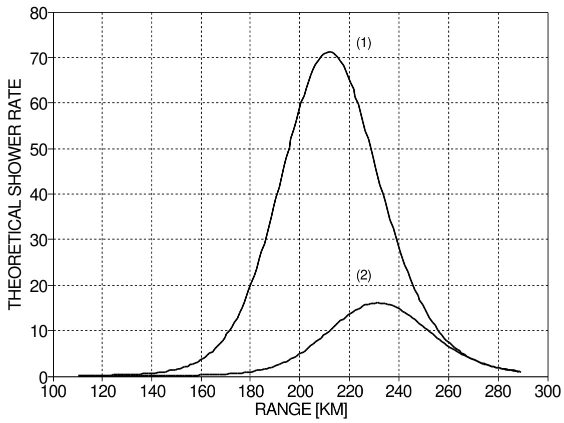

Figure 3. The theoretical range distribution as a function of atmospheric velocity, v, of meteor shower. The curve labelled by (1) corresponds to $v = 36\,\mathrm{km\,s^{-1}}$ while the one with (2) corresponds to $v = 72\,\mathrm{km\,s^{-1}}$. The computations were performed for the radiant of the Geminids.

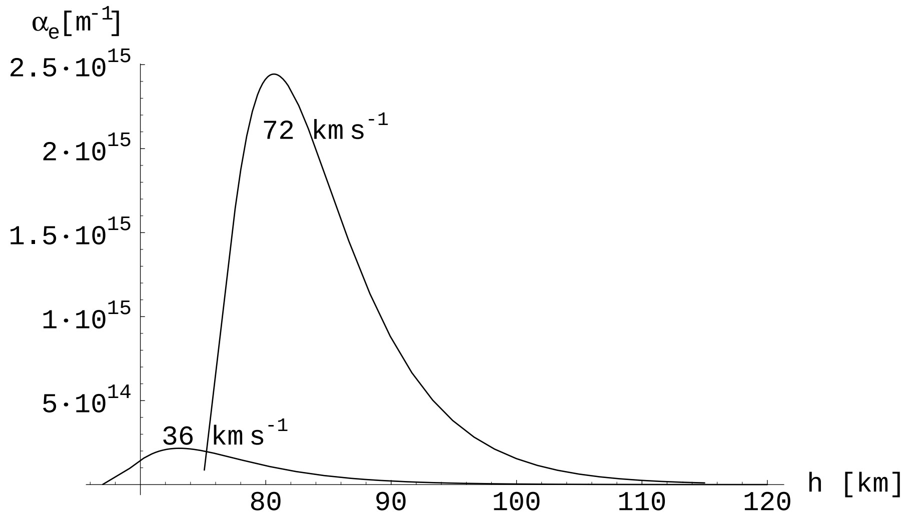

Figure 4. The theoretical ionization curve as a function of the atmospheric velocity v of meteor shower. Both curves that are marked with the corresponding value of v were computed for the radiant of Geminids between 1 and 2 UT, on the 13th of December, in 2000.

point of maximum ionization. The coordinates of shower radiants and velocities can be found either in Lovell (1954) or in Cook (1973).

5. Input data

As it was mentioned above, the Ondřejov meteor radar has been under operation since 1958. The unique long-term series of data have been managed to accumulate that are almost uninterrupted. The scientific research has mainly concentrated on four meteor showers: the Quadrantids, the Perseids, the Leonids and the Geminids. We tried to use the RaDiM for every year of each series. Besides, we also applied our method to two daytime showers that belong to the Taurid stream complex, ζ Perseids and β Taurids, observed in 2003 and to γ Draconid (Giacobinid) meteor shower observed during its last increased activity in 1998.

Despite the huge volume of data it was not easy at all to choose a suitable range distribution for computations belonging to a particular year. To make the previous statement clearer we have to mention a few fact about data processing. Because our observations are only single-station ones the method of observation does not permit to determine the direction in which a meteoroid plunges into the Earth's atmosphere. Hence, we do not know whether it belongs to the observed shower or to the background. To determine the shower activity we have to map also the level of background activity. For that reason an activity before and after shower activity has to be observed and after that we are able to construct a shower activity curve when subtracting the background rates from the ones gained during the shower activity period. This procedure was applied separately within each range distribution subinterval the whole distribution consisted of. When the background activity showed some unusual fluctuations we did not take this background into account and did not construct the range distribution at all. Moreover, badly determined background would influence the course of the particular range distribution namely in short distances by some strange fluctuations. The course of the range distribution of the sporadic meteors is of a different nature, the bigger number of echoes appears in shortest distances and their durations are rather short. When we looked for a range distribution suitable for computations by our method we met a lot of obstacles. We can divide them into two categories. The first one relates to technical problems such as interruptions of observations due to power supply failure, a very high noise level, human errors, problems with equipment and so on. Because of these technical faults it was not possible to determine the shower activity level and consequently to construct the relevant range distribution in some cases. Even in some years the observations were not performed at all due to reparations of the radar, its modernization or other technical problems. The second category includes problems with echo rates. It is clear that we need rather stronger activity in order the range distribution could be well-defined and the range distribution method could be applied. But, there was very low or even zero shower activity in some years, e. g., Quadrantids 1963, Perseids 1972. Thus, a majority of range distributions we have used were obtained during the maxima of shower activity and during larger time intervals, e. g., 2 hours. Furthermore, it was mentioned above that the Ondřejov meteor radar is able to observe unambiguously meteors within the range interval from 100 km to 600 km with the blocking gap between 300 and 360 km. Rich experience with observations and data processing indicates that an absolute majority of echoes occur within the interval $<100, 300>$ km. The shower rates in greater distances ($< 360, 600 >$ km) from the radar vary from one shower to another and do not exceed approximately 10%. This fact relates to the radar equation for overdense echoes (41) because the strength of signal decreases with the third power of distance from the radar. For that reason we have confined ourselves to range limits from 100 to 300 km. Moreover, the position of maximum of the range distribution changes with time due to time dependence of the shower radiant position as we have shown in Fig. 2. In view of this fact at some shower radiant position the maximum of the range distribution overstepped the range limits $< 100, 300 >$ km and was so badly determined that we were not again able to get anything. It was the case of faster meteor showers such as Leonids. But on the other hand, we have managed to construct two or more range distributions in one year under favorite conditions.

To sum up, searching for well-defined range distributions of overdense echoes was sometimes difficult. We proceeded as follows. Firstly, we usually divided shower rates into 20-km-wide or 25-km-wide intervals from 100 km to 300 km and then we interpolated them to get rates into 5-km-wide intervals with the assistance of the interpolating procedure SINOD (single interpolation in one dimension) published by Steffen (1990). The distributions obtained in this way served as an input to our computations. Sorting into 20 or 25 km wide intervals was a natural way how to make the observed data smoother. If the data were sorted directly into 5-km-wide intervals, smoothing would be very problematic in most cases due to fluctuations. Moreover, we eliminate in this way inaccuracy of determination of ranges from the film records which is about 2-3 km. The method of computations used was the Levenberg-Marquardt type (e. g., Press et al., 1992). Its brief outline is given in Appendix A. The procedure of its application was already described in Section 4.

The application of RaDiM to showers mentioned above and the relevant results will be described in subsequent articles.

6. Conclusion

The model we have developed provides us with possibility of the estimation of several important quantities connected with the structure of meteor showers and physical features of their forming meteoroids. We have developed the theory that makes use of the range distribution of shower meteors which have been observed by the Ondřejov meteor radar. Our approach to the construction of this theory is based on a simple physical theory of meteors with neglect of the deceleration of meteors contributing to the range distribution, which is justifiable. This distribution is a function of a few very important physical parameters characterizing the meteoroids of a particular shower such as the shape-density coefficient, K, and the ablation parameter, σ. Also the ionization coefficient, β, considered as a function of meteoroid velocity, is one of quantities our theoretical distribution depends on. The physical theory we have employed contains only the product $K \cdot \sigma$ in the final equations. Since observed meteoroids of all showers we have investigated are known to suffer from fragmentation during their atmospheric flights we tried to include this effect in our theory as well. It proved to be a rather tough proposition because to be able to consider the influence of fragmentation on the ionization curve, we have to know at what point of the curve the fragmentation takes place and its intensity. However, this is a piece of information which is not at our disposal in radar observations. Moreover, to obtain the ionization curve taking into account fragmentation we would have to sum up the signals of the parent body as well as of all fragments which is not possible to carry out in practice. On the other hand, it is clear that the influence of fragmentation manifests itself as shorter both light and ionization curves with their peaks being higher than the ones of nonfragmenting meteoroids. However, a very similar effect can be seen from the theory bearing in mind Levin's proposition about the variation of the meteoroid cross section, which is characterized by a new parameter, μ. Its classical value is 2/3. We have allowed it to vary within a much broader interval, $\mu > 0$. The value of $\mu \neq 2/3$ can also allow for different forms of ionization and light curves deviating from the one following from the simple classical theory of meteors.

Our principal formula of the range distribution (7) gives the number of meteors the radar in use can register within the collecting area of the echo plane. The older approach of Pecina (1982) to the determination of the domain of integration was based on the assumption that this is given by the point of maximum ionization. We have developed a more sophisticated approach in which we have abandoned that wrong assumption and our range distribution model relies only on the radar equation of overdense echoes.

Our theory allows us to compute two parameters related to the structure of meteor showers (and depending on solar longitude), Θ_{m_0} and s, and three quantities, $K \cdot \sigma$, μ and β, describing physical properties of meteoroids. We have managed to apply our model to 127 range distributions of 7 different showers. The relevant results will be published in subsequent articles.

At present there is no doubt that a majority of meteoroids are remnants of cometary nuclei and/or asteroids. Thus the investigation of meteoroid orbits and physical characteristics is important not only for meteor astronomy but also for the understanding of physical feature of the meteoroids parent bodies. The theory we have developed can be used to infer some physical parameters of shower meteors based on the radar observations. We hope that the range distribution model will become a handy tool enriching meteor astronomy.

Acknowledgements. This work has been supported by the Project AV0Z10030501 and partly by the grant No. 205/03/1405 of the Grant Agency of the Czech Republic.

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A. Mathematical method of getting parameters

We get the parameters $\Theta_{m_0}, s, a\,(K\cdot\sigma,\beta), b\,(K\cdot\sigma)$ and μ from the least-square fit of the theoretical rates computed according to our principal formula (7) to an observed range distribution:

$$\sum_{i=1}^{n}[N_i^O - N_i^C(\Theta_{m_0}, s, a(K\cdot\sigma,\beta), b(K\cdot\sigma),\mu)]^2 = \min. \tag{A1}$$

Here N_i^O is a number of meteors observed within a particular range interval and $N_i^C(\Theta_{m_0}, s, a(K\cdot\sigma,\beta), b(K\cdot\sigma))$ is a computed theoretical number of echoes. The symbol n stands for a total number of range intervals. Since N_i^C depends on all parameters except Θ_{m_0} in a nonlinear way we have to search for them iteratively. In computations like these, methods that take advantage of partial derivatives of N_i^C with respect to desired parameters proved to be useful. Let us give some indication of the iterative process. If we have to solve a task to look for unknown parameters p_j from the condition

$$Q(p_j) = \sum_{i=1}^{n} w_i[y_i - f_i(p_j)]^2 = \min\,, \tag{A2}$$

with w_i as apriori weights, y_i measured quantities and $f_i(p_j)$ their mathematical model that depend on p_j in a nonlinear way we approximate the function $Q(p_j)$ by its Taylor expansion around the parameters found in kth iteration:

$$\begin{aligned} Q(p_j^{k+1}) &= Q(p_j^k) + \sum_{\alpha=1}^{m}\frac{\partial Q}{\partial p_\alpha}(p_j^k)\,(p_\alpha^{k+1} - p_\alpha^k) \\ &+ \frac{1}{2}\sum_{\alpha,\beta=1}^{m}\frac{\partial^2 Q}{\partial p_\alpha \partial p_\beta}(p_j^k)\,(p_\alpha^{k+1} - p_\alpha^k)(p_\beta^{k+1} - p_\beta^k), \end{aligned} \tag{A3}$$

where m is a number of parameters the model depends on. At the minimum of (A3) the condition

$$\partial Q/\partial p_i^{k+1} = 0$$

should be satisfied. After taking the derivative of (A3) we get:

$$\frac{\partial Q}{\partial p_i}(p_j^k) + \sum_{\alpha=1}^{m}\frac{\partial^2 Q}{\partial p_i \partial p_\alpha}(p_j^k)\,(p_\alpha^{k+1} - p_\alpha^k) = 0. \tag{A4}$$

It is obvious that $\partial^2 Q/\partial p_i \partial p_\alpha$ are elements of the square matrix having dimension $m \times m$. After multiplication of (A4) by a matrix inverse to $\partial^2 Q/\partial p_i \partial p_\alpha$ we get the iterative recipe of the Gauss-Newton method (e. g. Meloun and Militký, 2005):

$$p_\gamma^{k+1} = p_\gamma^k - \sum_{\delta=1}^{m}\left(\frac{\partial^2 Q}{\partial p_\gamma \partial p_\delta}\right)^{-1}\frac{\partial Q}{\partial p_\delta}, \qquad \gamma = 1,\ldots,m. \tag{A5}$$

From (A2) it follows that

$$\frac{\partial Q}{\partial p_j} = -2\sum_{i=1}^{n} w_i\,[y_i - f_i(p_l)]\,\frac{\partial f_i}{\partial p_j}, \tag{A6}$$

and

$$\begin{aligned}\frac{\partial^2 Q}{\partial p_j \partial p_m} &= 2\sum_{i=1}^{n} w_i \frac{\partial f_i}{\partial p_j}\frac{\partial f_i}{\partial p_m} - 2\sum_{i=1}^{n} w_i \left[y_i - f_i(p_l)\right]\frac{\partial^2 f_i}{\partial p_j \partial p_m}\\ &\approx 2\sum_{i=1}^{n} w_i \frac{\partial f_i}{\partial p_j}\frac{\partial f_i}{\partial p_m}.\end{aligned}$$

The second term in the middle part of the last but one line is usually neglected because it causes the worse convergence behaviour during computations namely when the set of parameters is far from their values giving minimum of (A2). In order to simplify further expressions we define the vectors P_j, G_k and M_l together with the matrix S_{jk} by putting

$$P_j = \sum_{i=1}^{n} w_i \left[y_i - f_i\right]\frac{\partial f_i}{\partial p_j}, \quad S_{jk} = \sum_{i=1}^{n} w_i \frac{\partial f_i}{\partial p_j}\frac{\partial f_i}{\partial p_k}, \tag{A7}$$

$$G_k = \sum_{\alpha=1}^{m} S^{-1}_{k\alpha} P_\alpha, \qquad M_l = \sum_{\beta=1}^{m} S_{l\beta}\,\delta_{l\beta}\,G_\beta, \tag{A8}$$

where δ_{ij} stands for the Kronecker symbol. When comparing the definition of P_j and S_{jk} with partial derivatives of Q we can see that

$$\frac{\partial Q}{\partial p_\alpha} = -2\,P_\alpha, \ \frac{\partial^2 Q}{\partial p_\alpha \partial p_\beta} = 2\,S_{\alpha\beta}, \implies \left(\frac{\partial^2 Q}{\partial p_\alpha \partial p_\beta}\right)^{-1} = \frac{1}{2}\,S^{-1}_{\alpha\beta}.$$

Inserting for these derivatives into (A5) we already get the recipe for computation of p_j:

$$p_j^{k+1} = p_j^k + G_j. \tag{A9}$$

This is an explicit expression for the Gauss-Newton method.

The good initial estimate of searched parameters is necessary in order the Gauss-Newton method could work. If this is not the case the troubles when inverting the matrix $A_{\gamma\delta} \equiv \partial^2 Q/\partial p_\gamma \partial p_\delta$ are to be expected. To overcome these troubles Levenberg and Marquardt (e. g. Press et al., 1992) proposed to replace the matrix $A_{\gamma\delta}$ by another matrix $A_{\gamma\delta} + \lambda_{LM}\,\mathrm{diag}A_{\gamma\delta}$, where $\mathrm{diag}A_{\gamma\delta}$ is a diagonal matrix with elements coinciding with those of original $A_{\gamma\delta}$ and λ_{LM} is a parameter the authors recommend to choose in a rather artificial way. The sense of this proposition lies in the fact that the addition allows performing the inversion of the new matrix and the parameter λ_{LM} controls the length of an iteration step. The larger this parameter the shorter the step. This idea inspired us in further extension of this method. The length of an iteration step should depend on a quantitative expression of the fitting process. This is given by the magnitude of Q according to (A2). The greater Q the shorter should be the iteration length. This leads to a proposition to replace $A_{\gamma\delta}$ by $A_{\gamma\delta} + \lambda_P\,Q\,\mathrm{diag}A_{\gamma\delta}$. Inserting this matrix into (A5) and considering (A2) a function of p_j represented by the right hand side of (A5) the one-dimensional minimization of Q with respect to λ_P yields the desired expression for λ_P. Then the corresponding formula of a modified Levenberg-Marquardt method we have derived and made use can be written as

$$p_j^{k+1} = p_j^k + \sum_{k=1}^{m}\left(S_{jk} + \frac{\sum_{\alpha=1}^{m} M_\alpha G_\alpha}{\sum_{\beta,\gamma=1}^{m} S^{-1}_{\beta\gamma}\,M_\beta\,M_\gamma}\,\mathrm{diag}\,S_{jk}\right)^{-1} P_k. \tag{A10}$$

This approach is especially suitable when the normalized quantities in (A2) are used since the value of Q at minimum should be less than unity. We have used this recipe in all our computations. These begin with a chosen initial estimate. The subsequent values of parameters are computed using (A10). The computations are carried out until the subsequent sets of parameters differ by more than the prescribed constant.

To complete this chapter we have to add explicit expressions for N_i^C from (A1). This is given by (7). When substituting there for $m_\infty^{-1/3} \equiv x$ we obtain the desired expression

$$N_i^C = \Theta_{m_0} \int_{t_1}^{t_2} dt \int_{S_{col}} (m_0^{1/3}\, x)^{3(s-1)} dS. \tag{A11}$$

The derivative of N_i^C with respect to Θ_{m_0} is also easy to write:

$$\frac{\partial N_i^C}{\partial\, \Theta_{m_0}} = \int_{t_1}^{t_2} dt \int_{S_{col}} (m_0^{1/3}\, x)^{3(s-1)} dS. \tag{A12}$$

The derivative with respect to s can easily be performed, too. The result reads

$$\frac{\partial N_i^C}{\partial\, s} = 3\, \Theta_{m_0} \int_{t_1}^{t_2} dt \int_{S_{col}} (m_0^{1/3}\, x)^{3(s-1)} \ln{(m_0^{1/3}\, x)}\, dS. \tag{A13}$$

The derivatives with respect to remaining parameters can be written as

$$\frac{\partial N_i^C}{\partial\, p} = 3(s-1) \int_{t_1}^{t_2} dt \int_{S_{col}} (m_0^{1/3}\, x)^{3(s-1)}\, \frac{1}{x}\, \frac{\partial x}{\partial p}\, dS, \tag{A14}$$

where p stands for one of the parameters a, b, μ. The particular derivatives can be evaluated from (29) when defining an auxiliary quantity $\xi = b\, \varrho / \cos z_R$:

$$\begin{aligned}
\frac{1}{x}\, \frac{\partial x}{\partial a} &= \frac{1}{2a}\, \frac{1-(1-\mu)\, \xi\, x}{1-(1-3\mu/2)\, \xi\, x} \\
\frac{1}{x}\, \frac{\partial x}{\partial b} &= -\frac{1}{b}\, \frac{\mu\, \xi\, x}{1-(1-3\mu/2)\, \xi\, x} \\
\frac{1}{x}\, \frac{\partial x}{\partial \mu} &= \frac{1}{2(1-\mu)^2}\, \frac{\mu\, (1-\mu)\, \xi\, x}{1-(1-3\mu/2)\, \xi\, x} + \\
&+ \frac{1}{2(1-\mu)^2}\, \frac{\{1-(1-\mu)\, \xi\, x\}\, \ln{\{1-(1-\mu)\, \xi\, x\}}}{1-(1-3\mu/2)\, \xi\, x}
\end{aligned}$$