Solution of equations of the perturbed motion in the general three-bodies problem

N.A. Solovaya and E.M. Pittich

Astronomical Institute of the Slovak Academy of Sciences, Interplanetary Matter Division, Dúbravská cesta 9, 842 28 Bratislava, The Slovak Republic

Received: December 5, 2001

Abstract. The differential equations of the perturbed motion for the particular case of the nonrestricted three-bodies problem are received, using the classical method of variation of constants. As the perturbing function the terms of the third order in the Hamiltonian were used. The equations of the perturbed motion of the periastrons of the inner and outer orbits and the node were integrated by the iteration method. As the example the stellar system ξ UMa was used.

Key words: three-bodies problem – perturbations of third order – ξ UMa

1. Introduction

The present study is the solution of equations of the perturbed motion of three points in which the masses of the points are comparable and the ratio of the semi-major axis of their orbits is the small parameter. The solution of the intermediate orbits received by Hamilton-Jacobi's method (Orlov and Solovaya, 1988) is considered to be unperturbed and has the following form:

$$
\begin{aligned}
L_1 &= A_1\,, & B_1 &= \frac{\partial\varepsilon}{\partial A_1}(t-t_0) + l_1 + \frac{\partial W_1}{\partial A_1}\,, \\
L_2 &= A_2\,, & B_2 &= \frac{\partial\varepsilon}{\partial A_2}(t-t_0) + l_2 + \frac{\partial W_1}{\partial A_2}\,, \\
G_1 &= \frac{\partial W_1}{\partial g_1}\,, & B_3 &= \frac{\partial\varepsilon}{\partial A_3}(t-t_0) + \frac{\partial W_1}{\partial A_3}\,, \\
G_2 &= A_4\,, & B_4 &= \frac{\partial\varepsilon}{\partial A_4}(t-t_0) + g_2 + \frac{\partial W_1}{\partial A_4}\,, \\
c &= A_5\,, & B_5 &= \frac{\partial\varepsilon}{\partial A_5}(t-t_0) + h + \frac{\partial W_1}{\partial A_5}\,.
\end{aligned}
\tag{1}
$$

The general solution of the simplified equations of the unperturbed motion depends on ten arbitrary constants of integration A_i and B_i for $i = 1, 2, ...5$.

The connection between the constants ε and A_3 is expressed as:

$$\varepsilon = \frac{\gamma_1}{2\,A_1^2} + \frac{\gamma_2}{2\,A_2^2} - \frac{1}{16}\,\gamma_3\,\frac{A_1^4\,A_3}{A_2^3\,A_4^3}\,, \tag{2}$$

where

$$\gamma_1 = \frac{\beta_1^4}{\mu_1}, \qquad \gamma_2 = \frac{\beta_2^4}{\mu_2}, \qquad \gamma_3 = k^2\mu_1 m_2\frac{\beta_2^6}{\beta_1^4}\,,$$

$$\mu_1 = \frac{m_0\,m_1}{m_0 + m_1}\,, \qquad \mu_2 = \frac{(m_0 + m_1)\,m_2}{m_0 + m_1 + m_2}\,, \tag{3}$$

$$\beta_1 = k\frac{m_0 m_1}{\sqrt{m_0 + m_1}}, \qquad \beta_2 = k\frac{(m_0 + m_1)\,m_2}{\sqrt{m_0 + m_1 + m_2}}\,,$$

and k is the Gaussian constant and m_0, m_1, and m_2 are the masses of the points.

The Hamiltonian of this simplified dynamical system depends only on a single angular variable g_1 (the argument of the periastron of the close pair). $W_1 = W_1(g_1)$ is the seeking function, which defines the change of the eccentricity of a close orbit.

To obtain the approximate solution of the differential equations of the perturbed motion, the method of variation of constants was applied. The canonical constants A_i and B_i become the canonical variables.

The differential equations for A_i and B_i have the form:

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = \frac{\partial R}{\partial B_i}, \qquad \frac{\mathrm{d}B_i}{\mathrm{d}t} = -\frac{\partial R}{\partial A_i}, \tag{4}$$

where

$$R = F_3(g_1, g_2) + F_4(g_1, g_2) \tag{5}$$

As the perturbing function R we use the terms of the third $F_3(g_1, g_2)$ and the fourth $F_4(g_1, g_2)$ orders in the Hamiltonian, as determined by Solovaya and Pittich (1996). The terms of the third order were determined by the application of von Zeipel's method. The perturbing function depends on nonangular variables L_1, L_2, G_1, G_2 and two angular variables g_1 and g_2 – the periastrons of the both orbits.

L_j, G_j, and g_j are the canonical Delaunay elements ($j = 1$ for the inner orbit, $j = 2$ for the outer orbit). They can be expressed through the Keplerian elements as:

$$L_j = \beta_j\sqrt{a_j}, \qquad G_j = L_j\sqrt{1 - e_j^2}, \qquad g_j = \omega_j, \tag{6}$$

where

$$q = \frac{c^2 - G_1^2 - G_2^2}{2G_1G_2}, \tag{7}$$

The notations in the previous expressions have the usual meaning: a_j – the semi-major axis, e_j – the eccentricity, w_j – the arguments of the periastron. The eccentricities of both orbits can take any value from zero to one, c – the constant of the angular moment and q – cosine of the mutual inclination of the orbits. It can take any value from $0°$ to $180°$.

It may be remarked that the equations (4) become inconvenient to practical application because after differentiation of R with respect to A_i the time t appears as a factor in the coefficients of the periodic terms. Therefore, instead of the variables B_i, new functions were introduction (Pittich and Solovaya, 1998).

As the new variables the secular parts of the angular variables in the perturbed part of the Hamiltonian were taken. Denote them as λ_i. Using the Jacobi theorem about the canonical transformation we found the corresponding variables Λ_i.

The canonical system of the differential equations in the new variables can be written as:

$$\frac{\mathrm{d}\Lambda_i}{\mathrm{d}t} = \frac{\partial Z}{\partial \lambda_i}\,, \qquad \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = -\frac{\partial Z}{\partial \Lambda_i}\,, \tag{8}$$

for $i = 1, 2, ...5$, where $Z = \varepsilon + R$.

The Hamiltonian of the transformed system of the perturbed motion depends on ε and R. ε corresponds to the unperturbed motion. The second term R is the perturbing function.

2. The perturbing function R

The Hamiltonian of the transformed system of the differential equations of the perturbed motion equals the sum of the two terms: ε and R. The first term corresponds to unperturbed motion and when $R = 0$ will be constant.The perturbing function $R = F_3$ has the form:

$$F_3 = \frac{\gamma_3^2\, L_1^8}{1536\,\gamma_2\, G_2^7\, L_2^3} \left[F_{3_{sec}} + F_3(g_1) + F_3(g_1, g_2) + F_4(g_1, g_2)\right] , \tag{9}$$

where

$$F_{3_{sec}} = 2475\left(-1+\eta^2\right)^2 + \frac{45}{8}\left(85 - 210\,\eta^2 + 117\,\eta^4\right) e_2^2 +$$

$$+\frac{9}{2}\left(275 - 510\,\eta^2 + 243\,\eta^4\right)\frac{G_2}{L_2} +$$

$$
\begin{aligned}
&+36\,\eta\left(35-33\,\eta^2\right)\left(3+2\,e_2^2\right)\frac{G_2}{L_1}\,q+\\
&+\Bigl[18\left(-325+630\,\eta^2-309\,\eta^4\right)+\\
&+\frac{27}{4}\left(-175+470\,\eta^2-271\,\eta^4\right)e_2^2+\\
&+27\left(-125+210\,\eta^2-93\,\eta^4\right)\frac{G_2}{L_2}\Bigr]\,q^2+\\
&+36\,\eta\left(15-17\,\eta^2\right)\left(3+2\,e_2^2\right)\frac{G_2}{L_1}\,q^3+\\
&+\Bigl[45\left(75-110\,\eta^2+43\,\eta^4\right)+\\
&+\frac{45}{8}\left(-75+110\,\eta^2-43\,\eta^4\right)e_2^2+\\
&+\frac{81}{2}\left(75-110\,\eta^2+43\,\eta^4\right)\frac{G_2}{L_2}\Bigr]\,q^4\,,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
F_3(g_1) = &\left\{45\left(-5+8\,\eta^2-3\,\eta^4\right)\left(16+\frac{13}{2}\,e_2^2+6\,\frac{G_2}{L_2}\right)+\right.\\
&+540\,\eta\left(1-\eta^2\right)\left(3+2\,e_2^2\right)\frac{G_2}{L_1}\,q+\\
&+180\left(5-8\,\eta^2+3\,\eta^4\right)\left(9+e_2^2+6\,\frac{G_2}{L_2}\right)q^2+\\
&+540\,\eta\left(-1+\eta^2\right)\left(3+2\,e_2^2\right)\frac{G_2}{L_1}\,q^3+\\
&\left.+45\left(-5+8\,\eta^2-3\,\eta^4\right)\left(20+\frac{5}{2}\,e_2^2+18\frac{G_2}{L_2}\right)q^4\right\}\cos\left(2\,g_1\right)+\\
&+\frac{225}{2}\left(-1+\eta^2\right)^2\left\{\frac{5}{4}\left(8-e_2^2\right)+9\,\frac{G_2}{L_2}+\right.\\
&+\left[\frac{5}{2}\left(-8+e_2^2\right)-18\,\frac{G_2}{L_2}\right]q^2+\\
&\left.+\left[\frac{5}{4}\left(8-e_2^2\right)+9\,\frac{G_2}{L_2}\right]q^4\right\}\cos\left(4\,g_1\right),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
F_3(g_1, g_2) = {} & \frac{45\, e_1\, e_2\, \gamma_2\, \gamma_4\, G_2^2}{\gamma_3^2\, L_1^2} \times \\
& \times \left[\left(7 - 3\,\eta^2\right)\left(-1 - 11\,q + 5\,q^2 + 15\,q^3\right)\cos\left(g_1 - g_2\right) + \right. \\
& +35\left(1 - \eta^2\right)(1 - q)\left(1 + q^2\right)\cos\left(3\,g_1 - g_2\right) + \\
& + \left(7 - 3\,\eta^2\right)\left(-1 + 11\,q + 5\,q^2 - 15\,q^3\right)\cos\left(g_1 + g_2\right) + \\
& \left. + \, 35\left(1 - \eta^2\right)\left(-1 + q^2\right)(1 + q)\cos\left(3\,g_1 + g_2\right)\right] + \\
& + \frac{3375}{8}\left(-1 + \eta^2\right)^2 e_2^2\,(-1 + q)\,(1 + q)^3 \cos\left(4\,g_1 - 2\,g_2\right) + \\
& + \left\{\frac{45}{2}\left(-5 + 8\,\eta^2 - 3\,\eta^4\right) e_2^2 + 720\,\eta\left(-1 + \eta^2\right) e_2^2\,\frac{G_2}{L_1} + \right. \\
& + \left[\frac{585}{2}\left(5 - 8\,\eta^2 + 3\,\eta^4\right) e_2^2 + 270\,\eta\left(-1 + \eta^2\right) e_2^2\,\frac{G_2}{L_1}\right] q + \\
& + \left[315\left(5 - 8\,\eta^2 + 3\,\eta^4\right) e_2^2 + 1440\,\eta\left(1 - \eta^2\right) e_2^2\,\frac{G_2}{L_1}\right] q^2 + \\
& + \left[\frac{675}{2}\left(-5 + 8\,\eta^2 - 3\,\eta^4\right) e_2^2 + 810\,\eta\left(1 - \eta^2\right) e_2^2\,\frac{G_2}{L_1}\right] q^3 + \\
& \left. + \, \frac{675}{2}\left(-5 + 8\,\eta^2 - 3\,\eta^4\right) e_2^2\, q^4\right\}\cos\left(2\,g_1 - 2\,g_2\right) + \\
& + \left\{\frac{9}{4}\left(275 - 510\,\eta^2 + 243\,\eta^4\right) e_2^2 + \right. \\
& +108\,\eta\left(15 - 17\,\eta^2\right) e_2^2\,\frac{G_2}{L_1}\, q + \\
& +18\left(-175 + 270\,\eta^2 - 111\,\eta^4\right) e_2^2\, q^2 + \\
& +108\,\eta\left(-15 + 17\,\eta^2\right) e_2^2\,\frac{G_2}{L_1}\, q^3 + \\
& \left. + \, \frac{135}{4}\left(75 - 110\,\eta^2 + 43\,\eta^4\right) e_2^2\, q^4\right\}\cos\left(2\,g_2\right) + \\
& + \left\{\frac{45}{2}\left(-5 + 8\,\eta^2 - 3\,\eta^4\right) e_2^2 + 720\,\eta\left(1 - \eta^2\right) e_2^2\,\frac{G_2}{L_1} + \right. \\
& + \left[\frac{585}{2}\left(-5 + 8\,\eta^2 - 3\,\eta^4\right) e_2^2 + 270\,\eta\left(-1 + \eta^2\right) e_2^2\,\frac{G_2}{L_1}\right] q +
\end{aligned}
$$

$$+\left[315\left(5-8\,\eta^2+3\,\eta^4\right)e_2^2+1140\,\eta\left(1-\eta^2\right)e_2^2\,\frac{G_2}{L_1}\right]q^2+$$

$$+\left[\frac{675}{2}\left(5-8\,\eta^2+3\,\eta^4\right)e_2^2+810\,\eta\left(1-\eta^2\right)e_2^2\,\frac{G_2}{L_1}\right]q^3+$$

$$\left.+\,\frac{675}{2}\left(-5+8\,\eta^2-3\,\eta^4\right)e_2^2\,q^4\right\}\cos\left(2\,g_1+2\,g_2\right)+$$

$$+\frac{3375}{8}\left(-1+\eta^2\right)^2 e_2^2\left(-1+q\right)^3(1+q)\cos\left(4\,g_1+2\,g_2\right). \tag{12}$$

And

$$F_4(g_1,g_2)=\frac{9}{8192}\,\gamma_5\,\frac{L_1^8}{L_2^{10}}\,\frac{\sqrt{1-e_2^2}}{\left(-1+e_2\right)^4\left(1+e_2\right)^4}\ \times$$

$$\times\,\Big[\left(8+40\,e_1^2+15\,e_1^4\right)\left(2+3\,e_2^2\right)\left(3-30\,q^2+35\,q^4\right)+$$

$$+140\,e_1^2\left(2+e_1^2\right)\left(2+3\,e_2^2\right)\left(-1+8\,q^2-7\,q^4\right)\cos 2g_1+$$

$$+735\,e_1^4\left(2+3\,e_2^2\right)\left(-1+q^2\right)^2\cos 4g_1+$$

$$+735\,e1^4\,e_2^2\left(1-q\right)\left(1+q\right)^3\cos\left(4g_1-2g_2\right)+$$

$$+140\,e_1^2\left(2+e_1^2\right)e_2^2\left(1+q\right)^2\left(1-7\,q+7\,q^2\right)\cos\left(2g_1-2g_2\right)+$$

$$+10\left(8+40\,e_1^2+15\,e_1^4\right)e_2^2\left(-1+8\,q^2-7\,q^4\right)\cos 2g_2+$$

$$+140\,e_1^2\left(2+e_1^2\right)e_2^2\left(-1+q\right)^2\left(1+7\,q+7\,q^2\right)\cos\left(2g_1+2g_2\right)+$$

$$+\,735\,e_1^4\,e_2^2\left(1-q\right)^3\left(1+q\right)\cos\left(4g_1+2g_2\right)\Big]\ , \tag{13}$$

where $\eta=\sqrt{1-e_1^2}$.

The effect of R is at least an order of magnitude smaller than that due to ε. Define the order of the terms: the value ε is defined by formula (2). It consists of three terms. They have orders: zero, first and second. (Remember, that the small parameter is the ratio of the semi-major axis). The next term of the Hamiltonian F_3 will have an order not less than the third. Before the solution of the equation (8) we must express the perturbing function R through new variables Λ_i and λ_3 and λ_4. The new variables Λ_1,Λ_2,Λ_4 and Λ_5 coincide with the old, the substitution may be produced by simplifying $\Lambda_1 = A_1$, $\Lambda_2 = A_2$, $\Lambda_4 = A_4$, $\Lambda_5 = A_5$. The angular variables g_1 and g_2 in the intermediate motion are

$$g_1=\nu_3(t-t_0)+\Phi_1 \qquad g_2=\nu_4(t-t_0)+\Phi_2\,, \tag{14}$$

where ν_3 and ν_4 are the secular terms, Φ_1 and Φ_2 the periodic terms in the motion of the periastrons of the orbits.

Then in the new variables:

$$\lambda_3 = \lambda_{30} + \nu_3(t - t_0), \quad \lambda_4 = \lambda_{40} + \nu_4(t - t_0), \tag{15}$$

The complex problem is to express the perturbing function R through Λ_3, so for this it is necessary to solve the equation

$$\Lambda_3 = \frac{2}{\pi} W_1\left(g_1\right) . \tag{16}$$

However, we may obtaine such a system of the differential equations of the perturbed motion, wherein Λ_3 will not be the required function. We will take the functions A_1, A_2, A_3, A_4, A_5 and λ_3, λ_4.

The system of the differential equations will not be canonical, but the dependence of the right parts of these functions will be simplified further. In the future, we will use only the term of the third order, the term of the fourth order is very small. The perturbing function in the new variables has the form:

$$F_3 = \frac{\gamma_3^2\, A_1^8}{1536\, \gamma_2\, A_4^7\, A_2^3}\, [F_{3_{sec}} + F_3(\lambda_3) + F_3(\lambda_3, \lambda_4)], \tag{17}$$

where

$$\begin{aligned}
F_{3_{sec}} = {} & 2475\left(-1+\eta^2\right)^2 + \frac{45}{8}\left(85 - 210\,\eta^2 + 117\,\eta^4\right)\left(1 - \frac{A_4^2}{A_2^2}\right) + \\
& + \frac{9}{2}\left(275 - 510\,\eta^2 + 243\,\eta^4\right)\frac{A_4}{A_2} + \\
& + 36\,\eta\left(35 - 33\,\eta^2\right)\left(3 + 2\left(1 - \frac{A_4^2}{A_2^2}\right)\right)\frac{A_4}{A_1}\,q + \\
& + \Bigg[18\left(-325 + 630\,\eta^2 - 309\,\eta^4\right) + \\
& + \frac{27}{4}\left(-175 + 470\,\eta^2 - 271\,\eta^4\right)\left(1 - \frac{A_4^2}{A_2^2}\right) + \\
& + 27\left(-125 + 210\,\eta^2 - 93\,\eta^4\right)\frac{A_4}{A_2}\Bigg]\,q^2 + \\
& + 36\,\eta\left(15 - 17\,\eta^2\right)\left(3 + 2\left(1 - \frac{A_4^2}{A_2^2}\right)\right)\frac{A_4}{A_1}\,q^3 + \\
& + \Bigg[45\left(75 - 110\,\eta^2 + 43\,\eta^4\right) +
\end{aligned}$$

$$+\frac{45}{8}\left(-75+110\,\eta^2-43\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+$$

$$+\left.\frac{81}{2}\left(75-110\,\eta^2+43\,\eta^4\right)\frac{A_4}{A_2}\right]q^4\,, \tag{18}$$

$$F_3(\lambda_3)=\left\{45\left(-5+8\,\eta^2-3\,\eta^4\right)\left(16+\frac{13}{2}\left(1-\frac{A_4^2}{A_2^2}\right)+6\,\frac{A_4}{A_2}\right)+\right.$$

$$+540\,\eta\left(1-\eta^2\right)\left(3+2\left(1-\frac{A_4^2}{A_2^2}\right)\right)\frac{A_4}{A_1}\,q+$$

$$+180\left(5-8\,\eta^2+3\,\eta^4\right)\left(9+\left(1-\frac{A_4^2}{A_2^2}\right)+6\,\frac{A_4}{A_2}\right)q^2+$$

$$+540\,\eta\left(-1+\eta^2\right)\left(3+2\left(1-\frac{A_4^2}{A_2^2}\right)\right)\frac{A_4}{A_1}\,q^3+$$

$$\left.+\,45\left(-5+8\,\eta^2-3\,\eta^4\right)\left(20+\frac{5}{2}\left(1-\frac{A_4^2}{A_2^2}\right)+18\frac{A_4}{A_2}\right)q^4\right\}\times$$

$$\times\cos\left(2\,\lambda_3\right)+$$

$$+\frac{225}{2}\left(-1+\eta^2\right)^2\left\{\frac{5}{4}\left(8-\left(1-\frac{A_4^2}{A_2^2}\right)\right)+9\,\frac{A_4}{A_2}+\right.$$

$$+\left[\frac{5}{2}\left(-8+\left(1-\frac{A_4^2}{A_2^2}\right)\right)-18\,\frac{A_4}{A_2}\right]q^2+$$

$$\left.+\left[\frac{5}{4}\left(8-\left(1-\frac{A_4^2}{A_2^2}\right)\right)+9\,\frac{A_4}{A_2}\right]q^4\right\}\cos\left(4\,\lambda_3\right), \tag{19}$$

$$F_3(\lambda_3,\lambda_4)=\frac{45\,\sqrt{1-\eta^2}\,\sqrt{1-\frac{A_4^2}{A_2^2}}\,\gamma_2\,\gamma_4\,A_4^2}{\gamma_3^2\,A_1^2}\times$$

$$\times\left[\left(7-3\,\eta^2\right)\left(-1-11\,q+5\,q^2+15\,q^3\right)\cos\left(\lambda_3-\lambda_4\right)+\right.$$

$$+35\left(1-\eta^2\right)\left(1-q\right)\left(1+q^2\right)\cos\left(3\,\lambda_3-\lambda_4\right)+$$

$$+\left(7-3\,\eta^2\right)\left(-1+11\,q+5\,q^2-15\,q^3\right)\cos\left(\lambda_3+\lambda_4\right)+$$

$$\left.+\,35\left(1-\eta^2\right)\left(-1+q^2\right)\left(1+q\right)\cos\left(3\,\lambda_3+\lambda_4\right)\right]+$$

$$+\frac{3375}{8}\left(-1+\eta^2\right)^2\left(1-\frac{A_4^2}{A_2^2}\right)\left(-1+q\right)\left(1+q\right)^3\times$$

$$\times\cos\left(4\,\lambda_3-2\,\lambda_4\right)+$$

$$+\left\{\frac{45}{2}\left(-5+8\,\eta^2-3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+\right.$$

$$+720\,\eta\left(-1+\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}+$$

$$+\left[\frac{585}{2}\left(5-8\,\eta^2+3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+\right.$$

$$\left.+270\,\eta\left(-1+\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\right]q+$$

$$+\left[315\left(5-8\,\eta^2+3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+\right.$$

$$\left.+1440\,\eta\left(1-\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\right]q^2+$$

$$+\left[\frac{675}{2}\left(-5+8\,\eta^2-3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+\right.$$

$$\left.+810\,\eta\left(1-\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\right]q^3+$$

$$\left.+\frac{675}{2}\left(-5+8\,\eta^2-3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)q^4\right\}\cos\left(2\,\lambda_3-2\,\lambda_4\right)+$$

$$+\left\{\frac{9}{4}\left(275-510\,\eta^2+243\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+\right.$$

$$+108\,\eta\left(15-17\,\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\,q+$$

$$+18\left(-175+270\,\eta^2-111\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)q^2+$$

$$+108\,\eta\left(-15+17\,\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\,q^3+$$

$$+\frac{135}{4}\left(75-110\,\eta^2+43\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)q^4\Bigg\}\cos\left(2\,\lambda_4\right)+$$

$$+\Bigg\{\frac{45}{2}\left(-5+8\,\eta^2-3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+720\,\eta\left(1-\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}+$$

$$+\left[\frac{585}{2}\left(-5+8\,\eta^2-3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+270\,\eta\left(-1+\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\right]q+$$

$$+\left[315\left(5-8\,\eta^2+3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+1140\,\eta\left(1-\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\right]q^2+$$

$$+\left[\frac{675}{2}\left(5-8\,\eta^2+3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)+810\,\eta\left(1-\eta^2\right)\left(1-\frac{A_4^2}{A_2^2}\right)\frac{A_4}{A_1}\right]q^3+$$

$$+\frac{675}{2}\left(-5+8\,\eta^2-3\,\eta^4\right)\left(1-\frac{A_4^2}{A_2^2}\right)q^4\Bigg\}\cos\left(2\,\lambda_3+2\,\lambda_4\right)+$$

$$+\frac{3375}{8}\left(-1+\eta^2\right)^2\left(1-\frac{A_4^2}{A_2^2}\right)\left(-1+q\right)^3\left(1+q\right)\cos\left(4\,\lambda_3+2\,\lambda_4\right). \tag{20}$$

These functions may easily be differentiated. We must integrate the system of the differential equations

$$\frac{\mathrm{d}A_i}{\mathrm{d}t}=\frac{\partial F_3}{\partial \lambda_i}\,, \qquad \frac{\mathrm{d}\lambda_i}{\mathrm{d}t}=-\frac{\partial F_3}{\partial A_i}\,, \tag{21}$$

for $i=1,2,...5$.

3. Perturbations of the third order in the motion of periastrons and node

We will integrate the system of the differential equations using one of the classical methods, e.g. by the iteration method. Since the perturbing function F_3 is an order of magnitude less than in the intermediate motion, the changes of the elements will be slow, over considerable periods of time. To the first approximation, therefore, we may consider the right-hand side of the equations (21) to be a function only of time. To define the secular motion of the periastrons and node we obtained the following equations:

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}t}=-\left(\frac{6\,\pi\,\delta}{2\,A_1\,\overline{G}_2^2\,K\,\Sigma_1}\right)\frac{\partial R}{\partial A_3}\,,$$

$$\frac{\mathrm{d}\lambda_4}{\mathrm{d}t} = -\left[\frac{Q_4\,\Sigma_1 + Q_5\,\Sigma_2 + Q_6\,\Sigma_4 + Q_7\,\Sigma_5}{A_1\,\overline{G}_2^2\,\Sigma_1}\,6\,\delta\right]\frac{\partial R}{\partial A_3} - \frac{\partial R}{\partial A_4}\,,$$

$$\frac{\mathrm{d}\lambda_5}{\mathrm{d}t} = -\left[\frac{Q_8\,\Sigma_1 + Q_6\,\Sigma_4 - Q_7\,\Sigma_5}{A_1\,\overline{G}_2^2\,\Sigma_1}\,6\,\delta\right]\frac{\partial R}{\partial A_3}\,, \tag{22}$$

where $\lambda_3 = \lambda_{3_0} + \nu_3\,(t - t_0)$, $\lambda_4 = \lambda_{4_0} + \nu_4\,(t - t_0)$, and $\lambda_5 = \lambda_{5_0} + \nu_5\,(t - t_0)$.

The values δ, K, Σ_i, and the constant coefficients Q_i must be taken from formulae of the intermediate motion (Orlov and Solovaya, 1974).

For determining the secular perturbations of the third order $\Delta\nu_3$, $\Delta\nu_4$, and $\Delta\nu_5$, it is necessary to solve equations (22) taking as the perturbing function the secular term of the function F_3, which is expressed by formula (18). This function may be differentiated easily with respect to A_3 and A_4.

As an example, the system ξ Ursa Majoris, whose components move along short-period orbits was chosen. It's elements for epoch $T_0 = 1900.0$ (Heintz, 1967) are the folowing:

$$m_0 = 0.83\,, \qquad m_1 = 0.30\,, \qquad m_2 = 0.92\,.$$

inner orbit:	outer orbit:
$a_1 = 1.56$ AU	$a_2 = 19.46 AU$
$e_1 = 0.56$	$e_2 = 0.414$
$T_1 = 1935.410$	$T_2 = 1935.170$
$\omega_1 = 146.00^\circ$	$\omega_2 = 127.5^\circ$
$\Omega_1 = 326.00^\circ$	$\Omega_2 = 101.5^\circ$
$I_1 = 86.3^\circ$	$I_2 = 122.65^\circ$.

Note that using the only intermediate orbit we received the following results: the periastron of the close pair librate around 270° with amplitude $262.96^\circ < g_1 < 277.04^\circ$. The period of the doubled frequency of the libration is 2568.5 years. The mean motion of periastron of the outer orbit $\nu_4 = -14.97^\circ\mathrm{century}^{-1}$, the motion of node $\nu_5 = 14.56^\circ\mathrm{century}^{-1}$. The secular perturbations of the third order, using the formulae (22) are: $\Delta\nu_3 = -0.55^\circ\mathrm{century}^{-1}$, $\Delta\nu_4 = 0.45^\circ\mathrm{century}^{-1}$, $\Delta\nu_5 = -0.53^\circ\mathrm{century}^{-1}$. In the triple stellar systems, where masses are of the order of the solar mass, these additional values to the secular perturbations are small. But over a large time interval they must be taken into consideration.

4. The long-period perturbations of the third order in the eccentricity of the outer orbit

The perturbing function $F_3(g_1, g_2)$ depends on two angular variables g_1 and g_2 – the arguments of periastrons of the inner orbit (close pair) and the outer orbit (distant component). In the intermediate motion the eccentricity of the outer orbit e_2 is constant. But in the perturbed motion e_2 has the long-period perturbations. The previous formulae make it possible to receive the analytical expression for the long-period variations in the eccentricity e_2.

Take the differential equations from the formula (21):

$$\frac{\mathrm{d}A_4}{\mathrm{d}t} = \frac{\partial R}{\partial \lambda_4}\,. \tag{23}$$

The value A_4 may be easy expressed through the Keplerian elements:

$$A_4 = L_2 \sqrt{1 - e_2^2}\,. \tag{24}$$

Then

$$\frac{\mathrm{d}A_4}{\mathrm{d}t} = -A_2 \frac{e_2}{\sqrt{1 - e_2^2}} \frac{\mathrm{d}e_2}{\mathrm{d}t}\,. \tag{25}$$

The differential equation for e_2 has the form:

$$\frac{\mathrm{d}e_2}{\mathrm{d}t} = -\frac{\sqrt{1 - e_2^2}}{L_2\, e_2} \frac{\partial R}{\partial \lambda_4} = -\frac{A_4}{A_2^2\, e_2} \frac{\partial R}{\partial \lambda_4}\,, \tag{26}$$

where R is the perturbing function (20).

Then for the first approximation we have the expression:

$$\begin{aligned}
\Delta_1 e_2 = & -\frac{\gamma_3^2\, A_1^8}{1536\, \gamma_2\, A_4^6\, A_2^5} \Big\{ 9\, e_2 \left(-1 + q^2\right) \big[-275 + 510\, \eta^2 - \\
& -243\, \eta^4 + +48\, \eta \left(-15 + 17\, \eta^2\right) q \frac{A_4}{A_1} + \\
& + \left(1125 - 1650\, \eta^2 + 645\, \eta^4\right) q^2 \big] \frac{\cos\left(2\, t\, \nu_4 + \overline{\nu}_1\right)}{4\, \nu_4} + \\
& +3375 \left(-1 + \eta^2\right)^2 e_2 \left(1 - q\right) \left(1 + q\right)^3 \frac{\cos\left[2\, t \left(-2\, \nu_3 + \nu_4\right) + \overline{\nu}_2\right]}{8 \left(2\, \nu_3 - \nu_4\right)} + \\
& +3375 \left(-1 + \eta^2\right)^2 e_2 \left(-1 + q\right)^3 \left(1 + q\right) \frac{\cos\left[2\, t \left(2\, \nu_3 + \nu_4\right) + \overline{\nu}_3\right]}{8 \left(2\, \nu_3 + \nu_4\right)} + \\
& +45 \left(-1 + \eta^2\right) e_2 \Big[\left(-5 + 3\, \eta^2\right) \left(1 + q\right)^2 \left(1 - 15\, q + 15\, q^2\right) +
\end{aligned}$$

$$+4\,\eta\left(-8-3\,q+16\,q^2+9\,q^3\right)\frac{A_4}{A_1}\Bigg]\,\frac{\cos\left[2\,t\,(\nu_3-\nu_4)+\overline{\nu}_4\,\right]}{2\,(\nu_3-\nu_4)}+$$

$$+\frac{90\,e_1\,\gamma_2\,\gamma_4}{\gamma_3^2}\,q\,\frac{A_4^2}{A_2^2}\times$$

$$\times\Bigg[35\left(1-\eta^2\right)(-1+q)\,(1+q)^2\,\frac{\cos\left[t\,(-3\,\nu_3+\nu_4)+\overline{\nu}_5\,\right]}{2\,(3\,\nu_3-\nu_4)}+$$

$$+\left(-7+3\,\eta^2\right)\left(-1-11\,q+5\,q^2+15\,q^3\right)\frac{\cos\left[t\,(-\nu_3+\nu_4)+\overline{\nu}_6\,\right]}{2\,(\nu_3-\nu_4)}+$$

$$+35\left(1-\eta^2\right)(-1+q)^2\,(1+q)\,\frac{\cos\left[t\,(3\,\nu_3+\nu_4)+\overline{\nu}_7\,\right]}{2\,(3\,\nu_3+\nu_4)}+$$

$$+\left(-7+3\,\eta^2\right)\left(1-11\,q-5\,q^2+15\,q^3\right)\frac{\cos\left[t\,(\nu_3+\nu_4)+\overline{\nu}_8\,\right]}{2\,(\nu_3+\nu_4)}\Bigg]+$$

$$+45\left(-1+\eta^2\right)e_2^2\Bigg[\left(5-3\,\eta^2\right)(-1+q)^2\left(1+15\,q+15\,q^2\right)+$$

$$+4\,\eta\left(-8+3\,q+16\,q^2-9\,q^3\right)\frac{A_4}{A_1}\Bigg]\,\frac{\cos\left[2\,t\,(\nu_3+\nu_4)+\overline{\nu}_9\,\right]}{2\,(\nu_3+\nu_4)}\Bigg\}\,, \qquad (27)$$

where $\overline{\nu}_k$ is the combination of $\overline{\nu}_3$ and $\overline{\nu}_4$.

As illustration we present the long-period perturbations in the eccentricity of the outer orbits in the triple stellar system of ξ UMa (see Figs. 1–3).

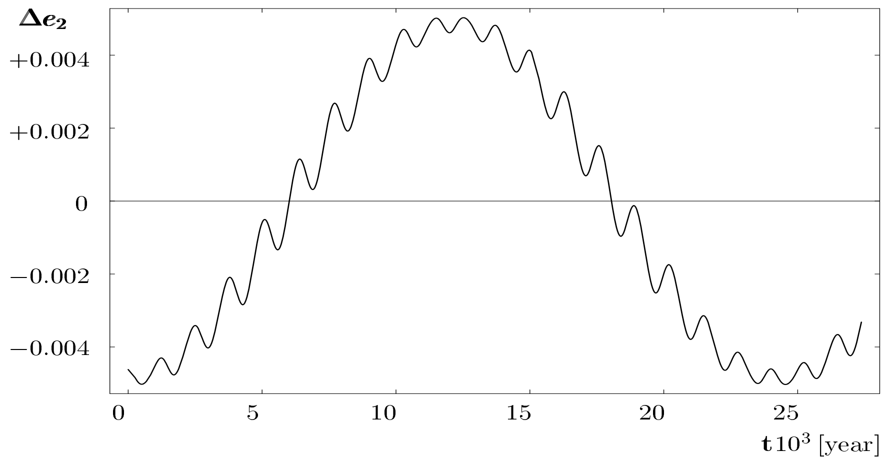

Figure 1. Long-period perturbations of the eccentricity of the outer orbit of ξ UMa within a period of 25 000 years. As the perturbing function $F_3(\lambda_3, \lambda_4)$ was used.

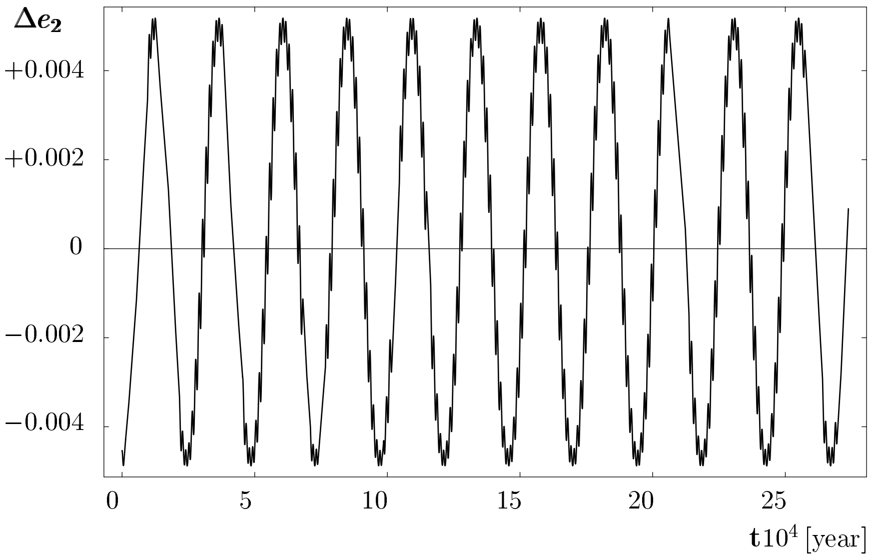

Figure 2. Long-period perturbations of the eccentricity of the outer orbit of ξ UMa within a period of 250 000 years. As the perturbing function $F_3(\lambda_3, \lambda_4)$ was used.

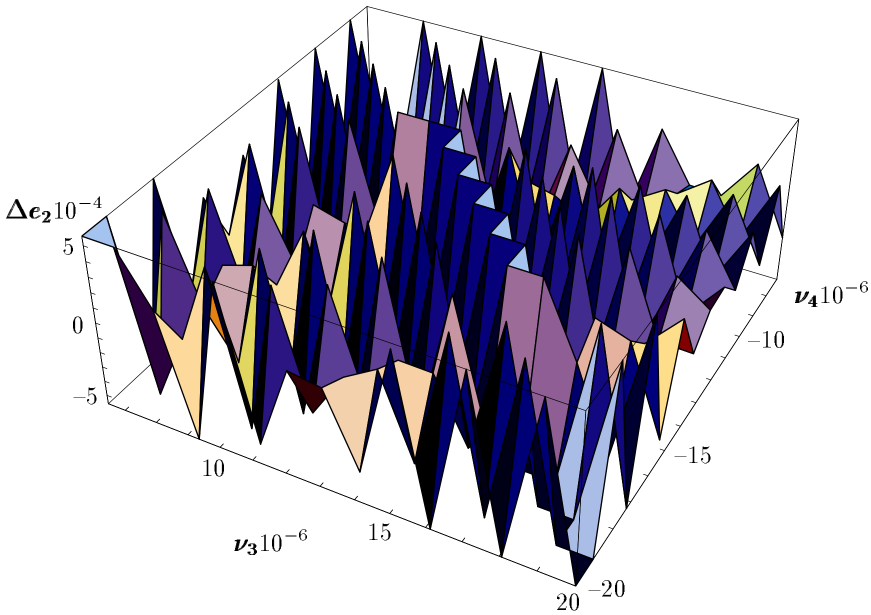

Figure 3. Long-period perturbations of the eccentricity of the outer orbit of ξ UMa with small secular perturbations in the motion of the periastrons of the inner and outer orbits.

5. Conclusions

We have derived the solution of equations of the perturbed motion of the nonrestricted three-bodies problem, using the method of variation of constants. The perturbing function take into account the terms of the third and the fourth orders in the Hamiltonian from which the short-period terms are excluded. The secular parts of the angular variables were taken as the new variables. The equations of the perturbed motion were integrated using the iteration method.

As an example, the real stellar system ξ UMa, whose components move along orbits with periods of 2 years (the close pair) and 60 years (the distant component), was presented. We see that in the triple stellar systems, where masses are of the order of the solar mass, the additional values to the secular perturbations and the long-period perturbations of the third order are small. However, for the precise reduction of observational data they must be taken into consideration.

Acknowledgements. This work was supported by the Slovak Academy of Sciences Grant No. 2/1005/21. The computations were done on ORIGIN 2000 computer of the Computer center of the Slovak Academy of Sciences.

References

Heints, W.: 1967, *Astron. Nachr.* **289**, 269

Orlov, A.A., N.A. Solovaya: 1974, *Trudy Moskovskogo Gosudarstvennogo Universiteta* **XLY**, 119

Orlov, A.A., N.A. Solovaya: 1988, in *The Few Body Problem*, ed.: M.J. Valtonen, Kluwer Academic Publishers, Dordrecht, The Netherlands, 243

Pittich, E.M., N.A. Solovaya: 1998, in *Fourth International Workshop on Positional Astronomy and Celestial Mechanics*, ed.: A. López García et al., Astron. Observ. Univ. of Valencia, Valencia, 109

Solovaya, N.A., E.M. Pittich: 1996, *Contrib. Astron. Obs. Skalnaté Pleso* **26**, 87