FROM REAL CAYLEY-DICKSON ALGEBRAS TO COMBINATORIAL GRASSMANNIANS

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Vienna University of Technology, Vienna, Austria
November 19, 2014
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- **$2^N$-nions, PG($N - 1, 2$), a binomial**
  
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- Acknowledgments
Introduction
Introduction: Cayley-Dickson algebras

(A vector space over the field $F$ is, loosely speaking, an additive group that is closed under scalar multiplication with elements in $F$. If a multiplication is defined for the elements of a vector space, and if the vector space is closed under this multiplication then it is called an algebra.)
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As it is well known, the (real) Cayley-Dickson algebras represent a nested sequence $A_0, A_1, A_2, \ldots, A_N, \ldots$ of $2^N$-dimensional (in general non-associative) $\mathbb{R}$-algebras with $A_N \subset A_{N+1}$, where $A_0 = \mathbb{R}$ and where for any $N \geq 0$ $A_{N+1}$ comprises all ordered pairs of elements from $A_N$ with conjugation defined by

\[(x, y)^* = (x^*, -y)\]  \hspace{1cm} (1)

and multiplication usually by

\[(x, y)(X, Y) = (xX - Yy^*, x^*Y + Xy).\]  \hspace{1cm} (2)
Introduction: Canonical basis

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The basis elements (or units) $e_0, e_1, e_2, \ldots, e_{2^{N+1}-1}$ of $A_{N+1}$, $e_0$ being the real basis element (identity), can, of course, be chosen in various ways.
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Our preference will be the canonical basis

$$
e_0 = (e_0, 0), \quad e_1 = (e_1, 0), \quad \ldots, \quad e_{2^N-1} = (e_{2^N-1}, 0),$$

$$
e_{2^N} = (0, e_0), \quad e_{2^N+1} = (0, e_1), \quad \ldots, \quad e_{2^{N+1}-1} = (0, e_{2^N-1}),$$

where, by abuse of notation, the same symbols are also used for the basis elements of $A_N$. 
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- it also implies that for both \( a \) and \( b \) being non-zero we have \( e_a e_b = \pm e_{a \oplus b} \), where the symbol ‘\( \oplus \)’ denotes ‘exclusive or’ of the binary representations of \( a \) and \( b \).
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From the above expressions and Eqs. (1) and (2) one can readily find the product of any two distinct units of $A_{N+1}$ if the multiplication properties of those of $A_N$ are given.
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Such products are usually expressed/presented in a tabular form, and we shall also follow this tradition here.
Introduction: Binary projective spaces

Given a multiplication table of $A_N$, $N \geq 2$, it can be verified that the $2^N - 1$ imaginaries $e_a$, $1 \leq a \leq 2^N - 1$, form $\binom{2^N - 1}{2}/3$ distinguished triples $\{e_a, e_b, e_c\}$ that satisfy equation

$$e_a e_b = \pm e_c,$$

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and where each unit is found to belong to $2^{N-1} - 1$ such sets.
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and where each unit is found to belong to $2^{N-1} - 1$ such sets.

Regarding the imaginaries as points and their distinguished triples as lines, one gets a point-line incidence geometry where every line has three points and through each point there pass $2^{N-1} - 1$ lines and which is isomorphic to $\text{PG}(N - 1, 2)$, the $(N - 1)$-dimensional projective space over the smallest Galois field $GF(2)$. 
Introduction: 2 types of lines

Let us assume, without loss of generality, that the elements in any distinguished triple \( \{ e_a, e_b, e_c \} \) of \( A_N \) are ordered in such a way that \( a < b < c \).
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Then, for \( N \geq 3 \), we can naturally speak about two different kinds of triples and, hence, two distinct kinds of lines of the associated \( 2^N \)-nionic PG\((N - 1, 2)\), according as \( a + b = c \) or \( a + b \neq c \); in what follows a line of the former/latter kind will be called ordinary/defective.
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This stratification of the line-set of the PG\((N - 1, 2)\) induces a similar partition of the point-set of the latter space into several types, where a point of a given type is characterized by the same number of lines of either kind that pass through it.
Introduction: Refined structure of our $\text{PG}(N - 1, 2)$

Obviously, if our projective space $\text{PG}(N - 1, 2)$ is regarded as an abstract geometry *per se*, every point and/or every line in it has *the same* footing.
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So, to account for the above-described ‘refinement’ of the structure of our $2^N$-nionic $\text{PG}(N - 1, 2)$, we need a representation of this space where each point/line is ascribed a certain ‘internal’ structure.
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To this end in view, we need to introduce a few notions/concepts from the realm of finite geometry.
Introduction: Configuration

We start with a finite point-line incidence structure $C = (\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P}$ and $\mathcal{L}$ are, respectively, finite sets of points and lines and where $I$ is a binary relation indicating which point-line pairs are incident.
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Here, we shall only be concerned with $\mathcal{C}$’s that are configurations. A $(v_r, b_k)$-configuration is a $\mathcal{C}$ where:
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if $v = b$ and $r = k$ we have a symmetric, or balanced, $(v, r)$-configuration.

A $(v, r, b, k)$-configuration with $v = \binom{r+k-1}{r}$ and $b = \binom{r+k-1}{k}$ is called a binomial configuration.
Introduction: Veldkamp space

A geometric hyperplane of \( C = (\mathcal{P}, \mathcal{L}, I) \) is a proper subset of \( \mathcal{P} \) such that a line from \( \mathcal{L} \)
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A geometric hyperplane of $\mathcal{C} = (\mathcal{P}, \mathcal{L}, I)$ is a proper subset of $\mathcal{P}$ such that a line from $\mathcal{L}$

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- a line of $\mathcal{V}(C)$ is the collection $H' H''$ of all geometric hyperplanes $H$ of $C$ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where $H'$ and $H''$ are distinct geometric hyperplanes.
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If each line of $C$ has three points and $C$ ‘behaves well,’ a line of $\mathcal{V}(C)$ is also of size three and can equivalently be defined as $\{H', H'', H' \Delta H''\}$. (Here $\Delta$ stands for the symmetric difference and an overbar denotes the complement of the object indicated.)
A combinatorial Grassmannian $G_k(|X|)$, where $k$ is a positive integer and $X$ is a finite set, is a point-line incidence structure whose
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If $|X| = N + 1$ and $k = 2$, then $G_2(N + 1)$ is a binomial

$$\left( \binom{N+1}{2}, \binom{N+1}{3} \right)$$-configuration.
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- $G_2(4)$ is the Pasch configuration,
- $G_2(5)$ is the Desargues configuration, and
- $G_2(N + 1)$’s, $N \geq 5$, are called *generalized* Desargues configurations.
Introduction: Objectives

From its definition it is obvious that $\mathcal{V}(C)$ is well suitable for our needs because its points, being themselves sets of points, have different ‘internal’ structure and so, in general, they can no longer be on the same par; clearly, the same applies to the lines $\mathcal{V}(C)$. 
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Our task thus basically boils down to finding such $C_N$ whose $\mathcal{V}(C_N)$ is isomorphic to $\text{PG}(N - 1, 2)$ and completely reproduces its $2^N$-nionic fine structure.
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Our task thus basically boils down to finding such $C_N$ whose $\mathcal{V}(C_N)$ is isomorphic to $\text{PG}(N - 1, 2)$ and completely reproduces its $2^N$-nionic fine structure.

This will be carried out in great detail for the first four non-trivial cases, $3 \leq N \leq 6$, which, when combined with the two trivial cases ($N = 1, 2$), will provide us with sufficient amount of information to spot a general pattern.
Octonions and the Pasch \((6_2, 4_3)\)-configuration
The smallest non-trivial case to be addressed is $A_3$, the algebra of octonions, whose multiplication table is given below.

**Table:** The multiplication table of the imaginary unit octonions $e_a$, $1 \leq a \leq 7$. For the sake of simplicity, in what follows we shall employ a short-hand notation $e_a \equiv a$; likewise for the real unit $e_0 \equiv 0$.

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</table>
Octonions: Fano plane

The above-given multiplication table implies the existence of the following seven \textit{distinguished} trios of imaginary units:

\begin{align*}
\{1, 2, 3\}, & \{1, 4, 5\}, \{1, 6, 7\}, \\
\{2, 4, 6\}, & \{2, 5, 7\}, \\
\{3, 4, 7\}, & \{3, 5, 6\}.
\end{align*}
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\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \\
\{2, 4, 6\}, \{2, 5, 7\}, \\
\{3, 4, 7\}, \{3, 5, 6\}.

Regarding the seven imaginary units as points and the seven distinguished triples of them as lines, we obtain a point-line incidence structure where each line has three points and, dually, each point is on three lines, and which is isomorphic to the smallest projective plane PG(2, 2), often called the Fano plane, depicted in the following figure:
Figure 1: An illustration of the structure of PG(2, 2), the Fano plane, that provides the multiplication law for octonions. The points of the plane are seven small circles. The lines are represented by triples of circles located on the sides of the triangle, on its altitudes, and by the triple lying on the big circle. The three imaginaries lying on the same line satisfy $e_a e_b = \pm e_c$. 
Octonions: Types of lines and points

It is then readily seen that we have six ordinary lines, namely

\[
\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\},
\]
Octonions: Types of lines and points

It is then readily seen that we have six ordinary lines, namely

\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\},

and only single defective one, viz.

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Similarly, our octonionic PG(2, 2) features two distinct types of points.
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Similarly, our octonionic PG(2, 2) features two distinct types of points. A type-one point is such that two lines passing through it are ordinary, the remaining one being defective; such a point lies in the set

\{3, 5, 6\} \equiv \alpha.
Octonions: Types of lines and points

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Similarly, our octonionic PG(2, 2) features two distinct types of points. A type-one point is such that two lines passing through it are ordinary, the remaining one being defective; such a point lies in the set

\{3, 5, 6\} \equiv \alpha.

A type-two point is such that every line passing through it is ordinary; such a point belongs to the set (gray color in the above figure)

\{1, 2, 4, 7\} \equiv \beta.
Octonions: Pasch configuration

A configuration $C_3$ whose Veldkamp space reproduces the above-described partitions of points and lines of $\text{PG}(2, 2)$ is the well-known \textit{Pasch} $(6_2, 4_3)$-configuration, $\mathcal{P}$. 
Octonions: Pasch configuration

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![Pasch configuration diagram]

Figure 2: An illustrative portrayal of the Pasch configuration: circles stand for its points, whereas its lines are represented by triples of points on common straight segments (three) and the triple lying on a big circle.
Octonions: Pasch configuration

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![Pasch configuration](image)

**Figure 2:** An illustrative portrayal of the Pasch configuration: circles stand for its points, whereas its lines are represented by triples of points on common straight segments (three) and the triple lying on a big circle.

This configuration also lives in the Fano plane and can be obtained from the latter by removal of any of its seven points and all the three lines passing through it.
Octonions: Veldkamp points

In order to see that $\mathcal{V}(\mathcal{P}) \cong \text{PG}(2, 2)$ we shall first show all seven geometric hyperplanes of $\mathcal{P}$:

![Diagram of hyperplanes labeled by imaginary units of octonions](image)

Figure 3: The hyperplanes are labeled by imaginary units of octonions in such a way that — as it is obvious from the next figure — the seven lines of the Veldkamp space of the Pasch configuration are identical with the seven distinguished triples of units, that is with the seven lines of our Fano plane.
We see that the hyperplanes are indeed of two different forms, of the required cardinalities three and four:
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- A member of the former set comprises two points at maximum distance from each other;
  such geometric hyperplane corresponds to a type-one (or \( \alpha \)-) point of \( \text{PG}(2, 2) \).
Octonions: Veldkamp points

We see that the hyperplanes are indeed of two different forms, of the required cardinalities three and four:

- A member of the former set comprises two points at maximum distance from each other; such geometric hyperplane corresponds to a type-one (or $\alpha$-) point of $\text{PG}(2, 2)$.
- A member of the latter set features three points on a common line; such geometric hyperplane corresponds to a type-two (or $\beta$-) point of our $\text{PG}(2, 2)$. 

Figure 4: A unified view of the seven Veldkamp lines of the Pasch configuration. For any three geometric hyperplanes lying on a given line of the Fano plane, one is the complement of the symmetric difference of the other two.
Sedenions and the Desargues $(10_3)$-configuration
**Sedenions: Multiplication table**

**Table:** The multiplication table of the imaginary unit sedenions \( e_a, 1 \leq a \leq 15. \)

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</table>
Sedenions: PG(3, 2)

An inspection of this table yields as many as 35 distinguished triples:

\[
\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{1, 8, 9\}, \{1, 10, 11\}, \{1, 12, 13\}, \{1, 14, 15\}, \\
\{2, 4, 6\}, \{2, 5, 7\}, \{2, 8, 10\}, \{2, 9, 11\}, \{2, 12, 14\}, \{2, 13, 15\}, \\
\{3, 4, 7\}, \{3, 5, 6\}, \{3, 8, 11\}, \{3, 9, 10\}, \{3, 12, 15\}, \{3, 13, 14\}, \\
\{4, 8, 12\}, \{4, 9, 13\}, \{4, 10, 14\}, \{4, 11, 15\}, \\
\{5, 8, 13\}, \{5, 9, 12\}, \{5, 10, 15\}, \{5, 11, 14\}, \\
\{6, 8, 14\}, \{6, 9, 15\}, \{6, 10, 12\}, \{6, 11, 13\}, \\
\{7, 8, 15\}, \{7, 9, 14\}, \{7, 10, 13\}, \{7, 11, 12\}.
\]
Sedenions: PG(3, 2)

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\[
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\{4, 8, 12\}, \{4, 9, 13\}, \{4, 10, 14\}, \{4, 11, 15\}, \\
\{5, 8, 13\}, \{5, 9, 12\}, \{5, 10, 15\}, \{5, 11, 14\}, \\
\{6, 8, 14\}, \{6, 9, 15\}, \{6, 10, 12\}, \{6, 11, 13\}, \\
\{7, 8, 15\}, \{7, 9, 14\}, \{7, 10, 13\}, \{7, 11, 12\}.
\]

Regarding the 15 imaginary units as points and the 35 distinguished trios of them as lines, we obtain a point-line incidence structure isomorphic to PG(3, 2), the smallest projective space — as depicted in the figure below.
Figure 5: A model of PG(3, 2) built, after Polster, around the pentagonal model of the generalized quadrangle of type GQ(2, 2) whose 15 lines are illustrated by triples of points lying on black line-segments (10 of them) and/or black arcs of circles (5). The remaining 20 lines of PG(3, 2) comprise four distinct orbits: the yellow, red, blue and green one consisting, respectively, of the yellow (\{1, 10, 11\}), red (\{1, 8, 9\}), blue (\{3, 13, 14\}) and green (\{3, 12, 15\}) line and other four lines we get from each by rotation through 72 degrees around the center of the pentagon.
Sedenions: PG(3, 2) – line types

25 ordinary lines:

\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{1, 8, 9\}, \{1, 10, 11\}, \{1, 12, 13\}, \{1, 14, 15\},
\{2, 4, 6\}, \{2, 5, 7\}, \{2, 8, 10\}, \{2, 9, 11\}, \{2, 12, 14\}, \{2, 13, 15\},
\{3, 4, 7\}, \{4, 8, 12\}, \{4, 9, 13\}, \{4, 10, 14\}, \{4, 11, 15\},
\{3, 8, 11\}, \{5, 8, 13\}, \{6, 8, 14\}, \{7, 8, 15\},
\{3, 12, 15\}, \{5, 10, 15\}, \{6, 9, 15\},
Sedenions: PG(3, 2) – line types

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\{3, 4, 7\}, \{4, 8, 12\}, \{4, 9, 13\}, \{4, 10, 14\}, \{4, 11, 15\},
\{3, 8, 11\}, \{5, 8, 13\}, \{6, 8, 14\}, \{7, 8, 15\},
\{3, 12, 15\}, \{5, 10, 15\}, \{6, 9, 15\},

10 defective ones:

\{3, 5, 6\}, \{3, 9, 10\}, \{3, 13, 14\},
\{5, 9, 12\}, \{5, 11, 14\},
\{6, 10, 12\}, \{6, 11, 13\},
\{7, 9, 14\}, \{7, 10, 13\}, \{7, 11, 12\}. 
Sedenions: PG(3, 2) – point types

Similarly, our sedenionic PG(3, 2) features two distinct types of points.
Sedenions: PG(3, 2) – point types

Similarly, our sedenionic PG(3, 2) features two distinct types of points.

A type-one point is such that four lines passing through it are ordinary, the remaining three being defective; such a point lies in the set

\[ \{3, 5, 6, 7, 9, 10, 11, 12, 13, 14\} \equiv \alpha. \]
Sedenions: PG(3, 2) – point types

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A type-one point is such that four lines passing through it are ordinary, the remaining three being defective; such a point lies in the set

\[ \{3, 5, 6, 7, 9, 10, 11, 12, 13, 14\} \equiv \alpha. \]

A type-two point is such that every line passing through it is ordinary; such a point belongs to the set

\[ \{1, 2, 4, 8, 15\} \equiv \beta, \]

being illustrated by gray shading in the above figure.
Sedenions: PG(3, 2) – refinement of line types

We see that all defective lines are of the same form, namely \{\alpha, \alpha, \alpha\}. 

Sedenions: $\text{PG}(3, 2)$ – refinement of line types

We see that all defective lines are of the same form, namely $\{\alpha, \alpha, \alpha\}$. The 25 ordinary lines split into two distinct families.
Sedenions: PG(3, 2) – refinement of line types

We see that all defective lines are of the same form, namely \(\{\alpha, \alpha, \alpha\}\). The 25 ordinary lines split into two distinct families. Ten of them are of the form \(\{\alpha, \beta, \beta\}\),

\[
\{1, 2, 3\}, \{1, 4, 5\}, \{1, 8, 9\}, \{1, 14, 15\},
\{2, 4, 6\}, \{2, 8, 10\}, \{2, 13, 15\},
\{4, 8, 12\}, \{4, 11, 15\},
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\]
Sedenions: PG(3, 2) – refinement of line types

We see that all defective lines are of the same form, namely \(\{\alpha, \alpha, \alpha\}\). The 25 ordinary lines split into two distinct families. Ten of them are of the form \(\{\alpha, \beta, \beta\}\),

\[
\{1, 2, 3\}, \{1, 4, 5\}, \{1, 8, 9\}, \{1, 14, 15\}, \\
\{2, 4, 6\}, \{2, 8, 10\}, \{2, 13, 15\}, \\
\{4, 8, 12\}, \{4, 11, 15\}, \\
\{7, 8, 15\},
\]

and the remaining 15 are of the form \(\{\alpha, \alpha, \beta\}\),

\[
\{1, 6, 7\}, \{1, 10, 11\}, \{1, 12, 13\}, \\
\{2, 5, 7\}, \{2, 9, 11\}, \{2, 12, 14\}, \\
\{3, 4, 7\}, \{4, 9, 13\}, \{4, 10, 14\}, \\
\{3, 8, 11\}, \{5, 8, 13\}, \{6, 8, 14\}, \\
\{3, 12, 15\}, \{5, 10, 15\}, \{6, 9, 15\}.
\]
Sedenions: Desargues configuration

A configuration $C_4$ whose Veldkamp space reproduces the above-described partitions of points and lines of $\text{PG}(3, 2)$ is the famous Desargues $(10_3)$-configuration, $\mathcal{D}$. 
Sedenions: Desargues configuration

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Figure 6: An illustrative portrayal of the Desargues configuration, built around the model of the Pasch configuration shown in Figure 2: circles stand for its points, whereas its lines are represented by triples of points on common straight segments (six), arcs of circles (three) and a big circle.
Figure 7: The fifteen geometric hyperplanes of the Desargues configuration.
Sedenions: Veldkamp points

We see that they are indeed of two different forms, and of required cardinality ten and five:
Sedenions: Veldkamp points

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- A member of the former set comprises a point and three points not collinear with it; such geometric hyperplane corresponds to a type-one (or $\alpha$-) point of $\text{PG}(3, 2)$.
Sedenions: Veldkamp points

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- A member of the former set comprises a point and three points not collinear with it; such geometric hyperplane corresponds to a type-one (or \( \alpha \)-) point of \( \text{PG}(3, 2) \).
- A member of the latter set features six points located on four lines, with two lines per each point, that is the Pasch configuration we introduced in the previous section; such a geometric hyperplane of \( D \) corresponds to a type-two (or \( \beta \)-) point of our \( \text{PG}(3, 2) \).
Sedenions: Veldkamp lines

It is also a straightforward task to verify that $\mathcal{V}(\mathcal{D})$ is endowed with 35 lines splitting into the required three families, as depicted in the next three figures.
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In these figures, the three geometric hyperplanes comprising a given Veldkamp line are distinguished by different colors, with their common elements being colored black.
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It is also a straightforward task to verify that $\mathcal{V}(\mathcal{D})$ is endowed with 35 lines splitting into the required three families, as depicted in the next three figures.

In these figures, the three geometric hyperplanes comprising a given Veldkamp line are distinguished by different colors, with their common elements being colored black.

For each Veldkamp line we also explicitly indicate its composition.
Sedenions: Veldkamp lines – defective

Figure 8: The ten Veldkamp lines of the Desargues configuration that represent the ten defective lines of the sedenionic $PG(3, 2)$. 
Sedenions: Veldkamp lines – ordinary ($\{\alpha, \beta, \beta\}$)

Figure 9: The ten Veldkamp lines of the Desargues configuration that represent the ten ordinary lines of the sedenionic PG(3, 2) of type $\{\alpha, \beta, \beta\}$. 
Figure 10: The fifteen Veldkamp lines of the Desargues configuration that represent the fifteen ordinary lines of the sedenionic $\text{PG}(3, 2)$ of type $\{\alpha, \alpha, \beta\}$.
Sedenions: Veldkamp space – ‘condensed’ view

Figure 11: A compact graphical view of illustrating the bijection between 15 imaginary unit sedenions and 15 geometric hyperplanes of the Desargues configuration, as well as between 35 distinguished triples of units and 35 Veldkamp lines of the Desargues configuration.
32-nions and the Cayley-Salmon $(15_4, 20_3)$-configuration
32-nions: $\text{PG}(4, 2)$

From the corresponding multiplication table (cf. arXiv:1405.6888) we infer the existence of 155 distinguished triples of imaginary units.
32-nions: PG(4, 2)

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Regarding the 31 imaginary units of $A_5$ as points and the 155 distinguished triples of them as lines, we obtain a point-line incidence structure isomorphic to PG(4, 2).
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Regarding the 31 imaginary units of $A_5$ as points and the 155 distinguished triples of them as lines, we obtain a point-line incidence structure isomorphic to PG(4, 2).

We next find that 65 lines of this space are defective and 90 ordinary.
32-nions: PG(4, 2) – point types

However, unlike the preceding two cases, there are *three* different types of points in our 32-nionic PG(4, 2):
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However, unlike the preceding two cases, there are three different types of points in our 32-nionic PG(4, 2):

- 10 $\alpha$-points, $\alpha \equiv \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28\}$, each of which is on 9 defective and 6 ordinary lines;
However, unlike the preceding two cases, there are three different types of points in our 32-nionic PG(4, 2):

- 10 $\alpha$-points, $\alpha \equiv \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28\}$, each of which is on 9 defective and 6 ordinary lines;
- 15 $\beta$-points, $\beta \equiv \{3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30\}$, each of which is on 7 defective and 8 ordinary lines; and
However, unlike the preceding two cases, there are three different types of points in our 32-nionic PG(4, 2):

- 10 \( \alpha \)-points, \( \alpha \equiv \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28\} \), each of which is on 9 defective and 6 ordinary lines;
- 15 \( \beta \)-points, \( \beta \equiv \{3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30\} \), each of which is on 7 defective and 8 ordinary lines; and
- 6 \( \gamma \)-points, \( \gamma \equiv \{1, 2, 4, 8, 16, 31\} \), each being on 15 ordinary (and, hence, on zero defective) lines.
32-nions: PG(4, 2) – line types

This stratification of the point-set of PG(4, 2) leads, in turn, to two different kinds of defective lines and three distinct kinds of ordinary lines.
32-nions: PG(4, 2) – line types

This stratification of the point-set of PG(4, 2) leads, in turn, to two different kinds of defective lines and three distinct kinds of ordinary lines. As per the defective lines, 45 of them are of type \( \{ \alpha, \alpha, \beta \} \),

\[
\{3, 13, 14\}, \{3, 21, 22\}, \{3, 25, 26\}, \\
\{5, 11, 14\}, \{5, 19, 22\}, \{5, 25, 28\}, \\
\{6, 11, 13\}, \{6, 19, 21\}, \{6, 26, 28\}, \\
\{7, 9, 14\}, \{7, 10, 13\}, \{7, 11, 12\}, \{7, 17, 22\}, \{7, 18, 21\}, \{7, 19, 20\}, \\
\{7, 25, 30\}, \{7, 26, 29\}, \{7, 27, 28\}, \\
\{9, 19, 26\}, \{9, 21, 28\}, \\
\{10, 19, 25\}, \{10, 22, 28\}, \\
\{11, 17, 26\}, \{11, 18, 25\}, \{11, 19, 24\}, \{11, 21, 30\}, \{11, 22, 29\}, \{11, 23, 28\}, \\
\{12, 21, 25\}, \{12, 22, 26\}, \\
\{13, 17, 28\}, \{13, 19, 30\}, \{13, 20, 25\}, \{13, 21, 24\}, \{13, 22, 27\}, \{13, 23, 26\}, \\
\{14, 18, 28\}, \{14, 19, 29\}, \{14, 20, 26\}, \{14, 21, 27\}, \{14, 22, 24\}, \{14, 23, 25\}, \\
\{15, 19, 28\}, \{15, 21, 26\}, \{15, 22, 25\},
\]
and 20 of type \{\beta, \beta, \beta\},

\{3, 5, 6\}, \{3, 9, 10\}, \{3, 17, 18\}, \{3, 29, 30\},

\{5, 9, 12\}, \{5, 17, 20\}, \{5, 27, 30\},

\{6, 10, 12\}, \{6, 18, 20\}, \{6, 27, 29\},

\{9, 17, 24\}, \{9, 23, 30\},

\{10, 18, 24\}, \{10, 23, 29\},

\{12, 20, 24\}, \{12, 23, 27\},

\{15, 17, 30\}, \{15, 18, 29\}, \{15, 20, 27\}, \{15, 23, 24\}.
Concerning the ordinary lines, one finds:

15 of them of type \{\beta, \beta, \beta\},

\{3, 12, 15\}, \{3, 20, 23\}, \{3, 24, 27\},
\{5, 10, 15\}, \{5, 18, 23\}, \{5, 24, 29\},
\{6, 9, 15\}, \{6, 17, 23\}, \{6, 24, 30\},
\{9, 18, 27\}, \{9, 20, 29\},
\{10, 17, 27\}, \{10, 20, 30\},
\{12, 17, 29\}, \{12, 18, 30\},
32-nions: \( \text{PG}(4, 2) \) – line types
60 of type \( \{ \alpha, \beta, \gamma \} \),

\[
\begin{align*}
\{1, 6, 7\}, \{1, 10, 11\}, \{1, 12, 13\}, \{1, 14, 15\}, \{1, 18, 19\}, \{1, 20, 21\}, \\
\{1, 22, 23\}, \{1, 24, 25\}, \{1, 26, 27\}, \{1, 28, 29\}, \\
\{2, 5, 7\}, \{2, 9, 11\}, \{2, 12, 14\}, \{2, 13, 15\}, \{2, 17, 19\}, \{2, 20, 22\}, \\
\{2, 21, 23\}, \{2, 24, 26\}, \{2, 25, 27\}, \{2, 28, 30\}, \\
\{3, 4, 7\}, \{3, 8, 11\}, \{3, 16, 19\}, \{3, 28, 31\}, \\
\{4, 9, 13\}, \{4, 10, 14\}, \{4, 11, 15\}, \{4, 17, 21\}, \{4, 18, 22\}, \\
\{4, 19, 23\}, \{4, 24, 28\}, \{4, 25, 29\}, \{4, 26, 30\}, \\
\{5, 8, 13\}, \{5, 16, 21\}, \{5, 26, 31\}, \\
\{6, 8, 14\}, \{6, 16, 22\}, \{6, 25, 31\}, \\
\{7, 8, 15\}, \{7, 16, 23\}, \{7, 24, 31\}, \\
\{8, 17, 25\}, \{8, 18, 26\}, \{8, 19, 27\}, \{8, 20, 28\}, \{8, 21, 29\}, \{8, 22, 30\}, \\
\{9, 16, 25\}, \{9, 22, 31\}, \\
\{10, 16, 26\}, \{10, 21, 31\}, \\
\{11, 16, 27\}, \{11, 20, 31\}, \\
\{12, 16, 28\}, \{12, 19, 31\}, \\
\{13, 16, 29\}, \{13, 18, 31\}, \\
\{14, 16, 30\}, \{14, 17, 31\},
\end{align*}
\]
and, finally,

15 of type \( \{\beta, \gamma, \gamma\} \), namely

\[
\{1, 2, 3\}, \{1, 4, 5\}, \{1, 8, 9\}, \{1, 16, 17\}, \{1, 30, 31\},
\{2, 4, 6\}, \{2, 8, 10\}, \{2, 16, 18\}, \{2, 29, 31\},
\{4, 8, 12\}, \{4, 16, 20\}, \{4, 27, 31\},
\{8, 16, 24\}, \{8, 23, 31\},
\{5, 16, 31\}.
\]
32-nions: a \((15_4, 20_3)\)-configuration

A \(C_5\) whose Veldkamp space accounts for this partitioning of both the point- and line-set of our 32-nionic \(\text{PG}(4, 2)\) is of type \((15_4, 20_3)\).
32-nions: a \((15_4, 20_3)\)-configuration

A \(C_5\) whose Veldkamp space accounts for this partitioning of both the point- and line-set of our 32-nionic \(\text{PG}(4, 2)\) is of type \((15_4, 20_3)\).

Figure 12: An illustration of the structure of the \((15_4, 20_3)\)-configuration, built around the model of the Desargues configuration shown in Figure 6. The five points added to the Desargues configuration are the three peripheral points and the red and blue point in the center. The ten lines added are three lines denoted by red color, three blue lines, three lines joining pairwise the three peripheral points and the line that comprises the three points in the center of the figure, that is the ones represented by a bigger red circle, a smaller blue circle and a medium-sized black one.
32-nions: Veldkamp points

This configuration possesses 31 distinct geometric hyperplanes that fall into three different types:
32-nions: Veldkamp points

This configuration possesses 31 distinct geometric hyperplanes that fall into three different types:

- A type-one hyperplane consists of a pair of skew lines at maximum distance from each other; there are 10 hyperplanes of this type and they correspond to $\alpha$-points of $\text{PG}(4, 2)$. 
32-nions: Veldkamp points

This configuration possesses 31 distinct geometric hyperplanes that fall into three different types:

- A type-one hyperplane consists of a pair of skew lines at maximum distance from each other; there are 10 hyperplanes of this type and they correspond to $\alpha$-points of PG(4, 2).

![Diagram of type-one hyperplanes](image)

Figure 13: The ten geometric hyperplanes of the (154, 203)-configuration of type one; the number below a subfigure indicates how many hyperplane’s copies we get by rotating the particular subfigure through 120 degrees around its center.
32-nions: Veldkamp points

- A type-two hyperplane features a point and all the points not collinear with it, the latter forming the Pasch configuration; there are 15 hyperplanes of this type, being counterparts of $\beta$-points of PG(4, 2).
32-nions: Veldkamp points

- A type-two hyperplane features a point and all the points not collinear with it, the latter forming the Pasch configuration; there are 15 hyperplanes of this type, being counterparts of $\beta$-points of PG(4, 2).

Figure 14: The fifteen geometric hyperplanes of the $(15_4, 20_3)$-configuration of type two.
32-nions: Veldkamp points

- A type-three hyperplane is identical with the Desargues configuration; we find altogether 6 guys of this type, each standing for a $\gamma$-point of $\text{PG}(4, 2)$. 
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Figure 15: The six geometric hyperplanes of the $(15_4, 20_3)$-configuration of type three.
32-nions: Veldkamp lines

Figure 16: The five types of Veldkamp lines of the \((15_4, 20_3)\)-configuration. Here, unlike Figures 8 to 10, each representative of a geometric hyperplane is drawn separately and different colors are used to distinguish between different hyperplane types: red is reserved for type one, yellow for type two and blue for type three hyperplanes. As before, black color denotes the core of a Veldkamp line, that is the elements common to all the three hyperplanes comprising it.
32-nions: Veldkamp lines

- Type-I (a): features two hyperplanes of type one and a type-two hyperplane and its core consists of two points that are at maximum distance from each other; there are \( \binom{10}{2} = 15 \times 6/2 = 45 \) Veldkamp lines of this type and they correspond to defective lines of PG(4, 2) of type \( \{\alpha, \alpha, \beta\} \).
32-nions: Veldkamp lines

- Type-I (a): features two hyperplanes of type one and a type-two hyperplane and its core consists of two points that are at maximum distance from each other; there are \( \binom{10}{2} = 15 \times 6/2 = 45 \) Veldkamp lines of this type and they correspond to defective lines of \( \text{PG}(4, 2) \) of type \( \{\alpha, \alpha, \beta\} \).

- Type-II (b): composed of three hyperplanes of type two that share three points on a common line; there are 20 Veldkamp lines of this type, having for their counterparts defective lines of \( \text{PG}(4, 2) \) of type \( \{\beta, \beta, \beta\} \).
32-nions: Veldkamp lines

- **Type-I (a):** features two hyperplanes of type one and a type-two hyperplane and its core consists of two points that are at maximum distance from each other; there are \( \binom{10}{2} = 15 \times 6/2 = 45 \) Veldkamp lines of this type and they correspond to defective lines of \( \text{PG}(4, 2) \) of type \( \{\alpha, \alpha, \beta\} \).

- **Type-II (b):** composed of three hyperplanes of type two that share three points on a common line; there are 20 Veldkamp lines of this type, having for their counterparts defective lines of \( \text{PG}(4, 2) \) of type \( \{\beta, \beta, \beta\} \).

- **Type-III (c):** also consists of three hyperplanes of type two, but in this case the three common points are pairwise at maximum distance from each other; there are 15 Veldkamp lines of this type, these being in a bijection with 15 ordinary lines of \( \text{PG}(4, 2) \) of type \( \{\beta, \beta, \beta\} \).
32-nions: Veldkamp lines

- Type-IV (d): exhibits a hyperplane of each type and its core is composed of a line and a point at the maximum distance from it; since for each line of our $(15_4, 20_3)$-configuration there are three points at maximum distance from it, there are $20 \times 3 = 60$ Veldkamp lines of this type, having their twins in ordinary lines of PG(4, 2) of type $\{\alpha, \beta, \gamma\}$. 
32-nions: Veldkamp lines

- **Type-IV (d):** exhibits a hyperplane of each type and its core is composed of a line and a point at the maximum distance from it; since for each line of our $(15_4, 20_3)$-configuration there are three points at maximum distance from it, there are $20 \times 3 = 60$ Veldkamp lines of this type, having their twins in ordinary lines of $\text{PG}(4, 2)$ of type $\{\alpha, \beta, \gamma\}$.

- **Type-V (e):** endowed with two hyperplanes of type three and a single one of type two, and its core is isomorphic to the Pasch configuration; hence, we have $\binom{6}{2} = 15$ Veldkamp lines of this type, being all representatives of ordinary lines of $\text{PG}(4, 2)$ of type $\{\beta, \gamma, \gamma\}$. 
This configuration can be viewed as *three* pairwise-disjoint triangles (r-b-y colors) in perspective from a line (gray), the centers of perspectivity of their pairs (bold black) being also on a common line; the two lines form a geometric hyperplane.
32-nions: \((15_4, 20_3)\)-configuration and GQ(2, 2)

- Given PG(3, 2) (i.e., a \((15_7, 35_3)\)-configuration) and any of 28 copies of GQ(2, 2) (i.e., a triangle-free \((15_3)\)-configuration) embedded in it; removing from the PG(3, 2) all the 15 lines of GQ(2, 2) we are left with nothing but a copy of our \((15_4, 20_3)\)-configuration!
32-nions: \((15_4, 20_3)\)-configuration and GQ(2, 2)

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- The point-line incidence geometry
  - whose points of are the 15 Pasch configurations of our \((15_4, 20_3)\)-configuration and
  - whose lines are triples of Pasch configurations which pairwise intersect in a single point

is isomorphic to GQ(2, 2)!
32-nions: *Hexagrammum mysticum*

Before embarking on the final case to be dealt with in detail, it is worth having a closer look at our \((15_4, 20_3)\)-configuration and point out its intimate relation with famous Pascal’s Mystic Hexagram.
32-nions: *Hexagrammum mysticum*

Before embarking on the final case to be dealt with in detail, it is worth having a closer look at our \((15_4, 20_3)\)-configuration and point out its intimate relation with famous Pascal’s Mystic Hexagram.

If six arbitrary points are chosen on a conic section and joined by line segments in any order to form a hexagon, then the three pairs of opposite sides of the hexagon meet in three points that lie on a straight line, the latter being called the Pascal line.
32-nions: *Hexagrammum mysticum*

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If six arbitrary points are chosen on a conic section and joined by line segments in any order to form a hexagon, then the three pairs of opposite sides of the hexagon meet in three points that lie on a straight line, the latter being called the Pascal line.

Taking the permutations of the six points, one obtains 60 different hexagons. Thus, the so-called complete Pascal hexagon determines altogether 60 Pascal lines, which generate a remarkable configuration of 146 points and 110 lines called the *hexagrammum mysticum*, or the complete Pascal figure.
32-nions: *Hexagrammum mysticum*

Both the points and lines of the complete Pascal figure split into several distinct families, usually named after their discoverers in the first half of the 19-th century.
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We are concerned here with the 15 Salmon points and the 20 Cayley lines which form a \((15_4, 20_3)\)-configuration. This configuration is discussed in some detail in M. Boben, G. Gévay and T. Pisanski, Danzer’s configuration revisited, preprint arXiv:1301.1067, where it is also depicted (Figure 6) and called the \textit{Cayley-Salmon} \((15_4, 20_3)\)-configuration.
32-nions: *Hexagrammum mysticum*

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And it is precisely this Cayley-Salmon \((15_4, 20_3)\)-configuration which our 32-nionic \((15_4, 20_3)\)-configuration is isomorphic to.
64-nions and a 
(21^5, 35^3)-configuration
64-nions: PG(5, 2)

From the corresponding multiplication table, which is freely available at http://jjj.de/tmp-zero-divisors/mult-table-64-ions.txt, we infer the existence of 651 distinguished triples of imaginary units.
64-nions: $\text{PG}(5, 2)$

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Regarding the 63 imaginary units of 64-nions as points and the 651 distinguished triples of them as lines, we obtain a point-line incidence structure where each line has three points and each point is on 31 lines, and which is isomorphic to $\text{PG}(5, 2)$. 
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Regarding the 63 imaginary units of 64-nions as points and the 651 distinguished triples of them as lines, we obtain a point-line incidence structure where each line has three points and each point is on 31 lines, and which is isomorphic to PG(5, 2).

Following the usual procedure, we find that 350 lines of this space are defective and 301 ordinary.
As in the preceding case, we encounter *three* different types of points in our 64-nionic PG(5, 2):
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- 35 $\alpha$-points, each of which is on 21 defective and 10 ordinary lines;
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- 35 $\alpha$-points, each of which is on 21 defective and 10 ordinary lines;
- 21 $\beta$-points, each of which is on 15 defective and 16 ordinary lines; and
- 7 $\gamma$-points, each being on 31 ordinary (and, hence, on zero defective) lines.
64-nions: PG(5, 2) – line types

This stratification of the point-set of PG(5, 2) leads, in turn, to
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64-nions: PG(5, 2) – line types

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- three different kinds of defective lines and
- four distinct kinds of ordinary lines.
64-nions: PG(5, 2) – line types

This stratification of the point-set of PG(5, 2) leads, in turn, to

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- four distinct kinds of ordinary lines.

Out of 350 defective lines, we find
64-nions: PG(5, 2) – line types

This stratification of the point-set of PG(5, 2) leads, in turn, to

- three different kinds of defective lines and
- four distinct kinds of ordinary lines.

Out of 350 defective lines, we find

- 105 of type \{\alpha, \alpha, \alpha\},
64-nions: PG(5, 2) – line types

This stratification of the point-set of PG(5, 2) leads, in turn, to

- three different kinds of defective lines and
- four distinct kinds of ordinary lines.

Out of 350 defective lines, we find

- 105 of type \{\alpha, \alpha, \alpha\},
- 210 of type \{\alpha, \alpha, \beta\} and
64-nions: PG(5, 2) – line types

This stratification of the point-set of PG(5, 2) leads, in turn, to
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- four distinct kinds of ordinary lines.

Out of 350 defective lines, we find
- 105 of type \(\{\alpha, \alpha, \alpha\}\),
- 210 of type \(\{\alpha, \alpha, \beta\}\) and
- 35 of type \(\{\beta, \beta, \beta\}\).
64-nions: $\text{PG}(5, 2)$ – line types

This stratification of the point-set of $\text{PG}(5, 2)$ leads, in turn, to

- three different kinds of defective lines and
- four distinct kinds of ordinary lines.

Out of 350 defective lines, we find

- 105 of type $\{\alpha, \alpha, \alpha\}$,
- 210 of type $\{\alpha, \alpha, \beta\}$, and
- 35 of type $\{\beta, \beta, \beta\}$.

On the other hand, 301 ordinary lines are partitioned into
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Out of 350 defective lines, we find
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- 35 of type \( \{\beta, \beta, \beta\} \).

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- 105 guys of type \( \{\alpha, \beta, \beta\} \),
64-nions: PG(5, 2) – line types

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- four distinct kinds of ordinary lines.

Out of 350 defective lines, we find

- 105 of type \{α, α, α\},
- 210 of type \{α, α, β\} and
- 35 of type \{β, β, β\}.

On the other hand, 301 ordinary lines are partitioned into

- 105 guys of type \{α, β, β\},
- 70 of type \{α, α, γ\},
64-nions: PG(5, 2) – line types

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- 210 of type \(\{\alpha, \alpha, \beta\}\) and
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- 70 of type \(\{\alpha, \alpha, \gamma\}\),
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- 105 of type \(\{\alpha, \beta, \gamma\}\) and
- 21 of type \(\{\beta, \gamma, \gamma\}\).
64-nions: \((21_5, 35_3)\)-configuration

The Veldkamp space mimicking such a fine structure of \(\text{PG}(5, 2)\) is that of a particular \((21_5, 35_3)\)-configuration.
64-nions: $(21_5, 35_3)$-configuration

The Veldkamp space mimicking such a fine structure of $\text{PG}(5, 2)$ is that of a particular $(21_5, 35_3)$-configuration.

To visualise this configuration, we build it around the model of the Cayley-Salmon $(15_4, 20_3)$-configuration of 32-nions.
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Given the Cayley-Salmon configuration, there are six points and 15 lines to be added to yield our \((21_5, 35_3)\)-configuration, and this is to be done in such a way that the configuration we started with forms a geometric hyperplane in it.
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Given the Cayley-Salmon configuration, there are six points and 15 lines to be added to yield our \((21_5, 35_3)\)-configuration, and this is to be done in such a way that the configuration we started with forms a geometric hyperplane in it.

As putting all the lines into a single figure would make the latter look rather messy, in the following we briefly illustrate this construction by drawing six different figures, each featuring all six additional points (gray) but only five out of 15 additional lines (these lines being also drawn in gray color), namely those passing through a selected additional point (represented by a doubled circle).
64-nions: \((21_5, 35_3)\)-configuration

Figure 17: An illustration of the structure of the \((21_5, 35_3)\)-configuration, built around the model of the Cayley-Salmon \((15_4, 20_3)\)-configuration shown in Figure 12.
64-nions: Veldkamp points

Employing this diagrammatical representation, one can verify that our $(21_5, 35_3)$-configuration exhibits 63 geometric hyperplanes that fall into three distinct types.
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- A type-one hyperplane consists of a line and its complement, which is the Pasch configuration; there are 35 distinct hyperplanes of this form, each corresponding to an \(\alpha\)-point of our PG(5, 2).
64-nions: Veldkamp points

Employing this diagrammatical representation, one can verify that our \((21_5, 35_3)\)-configuration exhibits 63 geometric hyperplanes that fall into three distinct types.

- A type-one hyperplane consists of a line and its complement, which is the Pasch configuration; there are 35 distinct hyperplanes of this form, each corresponding to an \(\alpha\)-point of our \(\text{PG}(5, 2)\).

- A type-two hyperplane comprises a point and its complement, which is the Desargues configuration; there are 21 hyperplanes of this form, each having a \(\beta\)-point for its \(\text{PG}(5, 2)\) counterpart.
64-nions: Veldkamp points

Employing this diagrammatical representation, one can verify that our $(21_5, 35_3)$-configuration exhibits 63 geometric hyperplanes that fall into three distinct types.

- A type-one hyperplane consists of a line and its complement, which is the Pasch configuration; there are 35 distinct hyperplanes of this form, each corresponding to an $\alpha$-point of our PG$(5, 2)$.

- A type-two hyperplane comprises a point and its complement, which is the Desargues configuration; there are 21 hyperplanes of this form, each having a $\beta$-point for its PG$(5, 2)$ counterpart.

- Finally, a type-three hyperplane is isomorphic to the Cayley-Salmon configuration; there are seven distinct guys of this type, each answering to a $\gamma$-point of the PG$(5, 2)$. 
64-nions: Veldkamp lines

We leave it with the interested reader to verify by themselves that the Veldkamp space of our $(21_5, 35_3)$-configuration indeed features 651 lines that do fall into the above-mentioned seven distinct kinds.
Figure 18: A ‘generalized Desargues’ view of the $(21_5, 35_3)$-configuration; four triangles (distinguished by different colors) are in perspective from a line (gray) in such a way that the points of perspectivity of six pairs of them form a Pasch configuration (black), the line and the Pasch configuration comprising a geometric hyperplane.
$2^N$-nions and a \[
\binom{N+1}{2}^N_{N-1}, \binom{N+1}{3}^3_{3}\]-configuration
$2^N$-nions: General pattern
If one also includes the trivial cases of
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- complex numbers ($N = 1$; the $(1, 0, 0)$-configuration, a point), and
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If one also includes the trivial cases of

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- quaternions ($N = 2$; the $(3_1, 1_3)$-configuration, a line),
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If one also includes the trivial cases of
- complex numbers ($N = 1$; the $(1_0, 0_3)$-configuration, a point), and
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we obtain the following sequence of configurations:
2^N-nions: General pattern

If one also includes the trivial cases of

- complex numbers \((N = 1; \text{the } (1_0, 0_3)-\text{configuration, a point}),\) and
- quaternions \((N = 2; \text{the } (3_1, 1_3)-\text{configuration, a line}),\)

we obtain the following sequence of configurations:

\((1_0, 0_3),\)
\((3_1, 1_3),\)
\((6_2, 4_3),\)
\((10_3, 10_3),\)
\((15_4, 20_3),\)
\((21_5, 35_3),\)

\(\ldots,\)

\(\binom{N + 1}{2}_{N-1}, \binom{N + 1}{3}_3,\)

\(\ldots.\)
$2^N$-nions: nesting – configurations

In other words, we get a nested sequence of \textit{binomial} $\left( \binom{r+k-1}{r} \right)_r, \left( \binom{r+k-1}{k} \right)_k$-configurations with $r = N - 1$ and $k = 3$. 
\(2^N\)-nions: nesting – configurations

In other words, we get a nested sequence of binomial
\(((r+k-1)_r, (r+k-1)_k)^r\)-configurations with \(r = N - 1\) and \(k = 3\).

Figure 19: A nested hierarchy of finite \((\binom{N+1}{2}_{N-1}, \binom{N+1}{3}_3)\)-configurations of \(2^N\)-nions for \(1 \leq N \leq 5\) when embedded in the Cayley-Salmon configuration \((N = 5)\).
$2^N$-nions: nesting – geometric hyperplanes

Denoting our generic $\binom{N+1}{2}^{N-1}, \binom{N+1}{3}^3$-configuration by $C_N$, we can express the types of geometric hyperplanes of the above-discussed cases in a compact form as follows

\begin{align*}
C_1 & : \emptyset, \\
C_2 & : C_1, \\
C_3 & : C_2, C_1 \sqcup C_1, \\
C_4 & : C_3, C_2 \sqcup C_1, \\
C_5 & : C_4, C_3 \sqcup C_1, C_2 \sqcup C_2, \\
C_6 & : C_5, C_4 \sqcup C_1, C_3 \sqcup C_2,
\end{align*}
This implies the following generic hyperplane compositions

\[ C_N: \ C_{N-1}, \ C_{N-2} \sqcup C_1, \ C_{N-3} \sqcup C_2, \ldots, \ C_{\frac{N}{2}} \sqcup C_{\frac{N}{2}-1}, \]

or

\[ C_N: \ C_{N-1}, \ C_{N-2} \sqcup C_1, \ C_{N-3} \sqcup C_2, \ldots, \ C_{\lfloor \frac{N}{2} \rfloor} \sqcup C_{\lfloor \frac{N}{2} \rfloor}, \]

according as \( N \) is, respectively, even or odd; here, the symbol ‘\( \sqcup \)’ stands for a disjoint union of two sets.
$2^N$-nions: combinatorial Grassmannians

A *combinatorial* Grassmannian $G_k(|X|)$, where $k$ is a positive integer and $X$ is a finite set, is a point-line incidence structure whose
2^N-nions: combinatorial Grassmannians

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- points are $k$-element subsets of $X$ and
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A *combinatorial* Grassmannian $G_k(|X|)$, where $k$ is a positive integer and $X$ is a finite set, is a point-line incidence structure whose

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- lines are $(k + 1)$-element subsets of $X$, incidence being inclusion.

If $|X| = N + 1$ and $k = 2$, then $G_2(N + 1)$ is a binomial $(\binom{N+1}{2}_{N-1}, \binom{N+1}{3}_3)$-configuration.
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\[ \left( \binom{N+1}{2} \right)_{N-1}, \left( \binom{N+1}{3} \right)_3 \]-configuration.

In particular,
$2^N$-nions: combinatorial Grassmannians

A combinatorial Grassmannian $G_k(\|X\|)$, where $k$ is a positive integer and $X$ is a finite set, is a point-line incidence structure whose

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- $G_2(4)$ is the Pasch configuration,
$2^N$-nions: combinatorial Grassmannians

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\left(\binom{N+1}{2}, \binom{N+1}{3}\right)-configuration.
\]

In particular,

- $G_2(4)$ is the Pasch configuration,
- $G_2(5)$ is the Desargues configuration, and...
$2^N$-nions: combinatorial Grassmannians

A *combinatorial* Grassmannian $G_k(|X|)$, where $k$ is a positive integer and $X$ is a finite set, is a point-line incidence structure whose

- points are $k$-element subsets of $X$ and
- lines are $(k + 1)$-element subsets of $X$,

incidence being inclusion.

If $|X| = N + 1$ and $k = 2$, then $G_2(N + 1)$ is a binomial

\[
\binom{N+1}{2} N - 1, \binom{N+1}{3} 3
\]

-configuration.

In particular,

- $G_2(4)$ is the Pasch configuration,
- $G_2(5)$ is the Desargues configuration, and
- $G_2(N + 1)$'s, $N \geq 5$, are called *generalized* Desargues configurations.
From our detailed examination of the four cases it follows that

- $C_3$, $C_4$, $C_5$, $C_6$, \ldots, $C_N$

are endowed with

- 1, 5, 15, 35, \ldots, $\binom{N+1}{4}$

Pasch configurations.
$2^N$-nions: combinatorial Grassmannians

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Pasch configurations.

As $\binom{N+1}{4}$ is also the number of Pasch configurations in $G_2(N+1)$, $N \geq 3$, we are thus naturally led to conjecture that

$$C_N \cong G_2(N + 1)$$
$2^N$-nions: combinatorial Grassmannians

This isomorphism indeed holds for $1 \leq N \leq 6$; for $N = 5$ and $N = 6$ it is illustrated in the following figure:
$2^N$-nions: combinatorial Grassmannians

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Figure 20: **Left:** – A diagrammatical proof of the isomorphism between $C_5$ and $G_2(6)$. The points of $C_5$ are labeled by pairs of elements from the set \{1, 2, \ldots , 6\} in such a way that each line of the configuration is indeed of the form \{\{a, b\}, \{a, c\}, \{b, c\}\}, $a \neq b \neq c \neq a$. **Right:** – A pictorial illustration of $C_6 \cong G_2(7)$. Here, the labels of six additional points are only depicted, the rest of the labeling being identical to that shown in the left-hand side figure.
Conclusion
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- An intriguing finite-geometrical underpinning of the multiplication tables of Cayley-Dickson algebras $A_N$, $3 \leq N \leq 6$, has been found that admits generalization to any higher-dimensional $A_N$. 
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Conclusion

- An intriguing finite-geometrical underpinning of the multiplication tables of Cayley-Dickson algebras $A_N$, $3 \leq N \leq 6$, has been found that admits generalization to any higher-dimensional $A_N$.
- This started with an observation that the multiplication properties of imaginary units of the algebra $A_N$ are encoded in the structure of the projective space $PG(N - 1, 2)$.
- Next, this space was shown to possess a refined structure stemming from particular properties of triples of imaginary units forming its lines.
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- This started with an observation that the multiplication properties of imaginary units of the algebra $A_N$ are encoded in the structure of the projective space $\text{PG}(N - 1, 2)$.

- Next, this space was shown to possess a refined structure stemming from particular properties of triples of imaginary units forming its lines.

- To account for this refinement, we employed the concept of the Veldkamp space of a point-line incidence structure $C_N$. 
Conclusion
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  - $C_6$ (64-nions) was shown to be identical with a particular $(21_5, 35_3)$-configuration that can be viewed as four triangles in perspective from a line where the points of perspectivity of six pairs of them form a Pasch configuration.
Conclusions

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- Finally, we conjectured that $C_N \cong G_2(N + 1)$, the later object being a combinatorial Grassmannian.
Acknowledgments

This work was partially supported by

- the VEGA Grant Agency, Project 2/0003/13, as well as by
- the Austrian Science Fund (FWF), Research Project M1564–N27.

Our sincere gratitude also goes to

- Hans Havlicek (Vienna University of Technology) for a number of critical remarks on the first draft of the paper,
- Jörg Arndt who computed for us multiplication tables of octonions, sedenions, 32-nions and 64-nions, and
- Boris Odehnal (University of Applied Arts, Vienna) for creating nice computer versions of several figures.