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The calculation of normal stars structure within the generalized polytropic model

M. Vavrukh and D. Dzikovskyi

Department of Astrophysics, Ivan Franko National University, Kyrylo & Methodiy str. 8, 79005 Lviv, Ukraine (E-mail: mvavrukh@gmail.com)

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Abstract. The generalized polytropic equation of state $P(\mathbf{r}) = K\big(\rho(\mathbf{r})/f(\mathbf{r})\big)^{4/3}$ for calculation of stars' characteristics of an arbitrary age with a spatially heterogeneous distribution of chemical composition ($f(\mathbf{r}) = \mu(\mathbf{r})/\bar{\mu}$, where $\mu(\mathbf{r})$ is the local value of the dimensionless molecular weight, and $\bar{\mu}$ is its average value over the star volume) was constructed by the Eddington method. Using the example of the Sun, it is shown that the standard polytropic model ($f(\mathbf{r}) = 1$) corresponds to the stars of zero age. The characteristics of the Sun in the modern epoch and their evolutionary changes were calculated. The obtained results are close to those based on the system of Schwarzschild equations. The proposed approach is applied to the calculation of the internal structure of the star model with axial rotation.

Key words: methods: analytical – stars: fundamental parameters – stars: interiors – stars: evolution

1. Introduction

Axial rotation of stars is an attribute of their existence. Traditionally, the influence of axial rotation on stars' characteristics is taken into account within the perturbation theory. Herewith, the zero approximation is a well-known polytropic model with spatially homogeneous chemical composition and equation of state $P(\mathbf{r}) = K[\rho(\mathbf{r})]^{1+1/n}$, where n and K are constants. In the particular case $n = 3$, this model was substantiated by Eddington at the beginning of the last century, when there was no information about the energy sources of stars and changes of chemical composition with the age of a star. The polytropic models with $n \neq 3$ have a phenomenological character. The theory of rotational polytropes was developed by Milne (1923), Chandrasekhar (1933), James (1964) and other researchers in the first half of the last century within the mechanical equilibrium equation. In the first of the two named works, the influence of rotation was taken into account as a perturbation and analytical expansions were used with accuracy to the square of angular velocity. The work of James (1964) is based on the numerical integration of the equilibrium equation.

The second stage in the development of the internal structure of stars, initiated in the middle of the last century, was based on the system of equations (Schwarzschild, 1958). However, in the study of solar structure and the evolution of stars with different masses (Eddington, 1988), there were employed spherically symmetrical models and rotation considered as a secondary factor. At the same time, Monaghan & Roxburgh (1965), Caimmi (1980), and Williams (1988) improved the semi-analytical theory of rotational polytropes with a spatially uniform chemical composition, corresponding to stars of zero age.

Already in the XXI-th century, a new direction of research has been arisen – the calculation of the internal structure of specific stars with high angular velocity. In particular, Kong et al. (2015) and Knopik et al. (2017), who focus on the internal structure of the star α Eri with angular velocity $\omega \approx 3 \cdot 10^{-5}\mathrm{s}^{-1}$, employed the computer 3D integration. However, the standard polytropic equation of state with index $n = 1$ reduces the persuasiveness of the obtained results for a star that is at the final stage of its evolution. The strict approach to the calculation of the internal structure of a star should be based on a system of differential equations. As it is known, the system of Schwarzschild equations for the model with a spherically symmetry consist of four ordinary differential equations. Taking into account the rotation leads to eight differential equations, which greatly complicated calculations. We propose a different approach to the calculation of characteristics of stars of different age at the presence of rotation. We find an approximate semi-analytical solutions of the mechanical equilibrium equation with rotation within a generalized equation of state, that allows us to take into account the spatially heterogeneous chemical composition, which arose as a result of thermonuclear reactions and corresponds to the star of certain age. Undoubtedly, such approach has a qualitative nature, but it greatly simplifies the calculations and plays the role of some express analysis for the selection of objects for the purpose of more detailed research.

In Section 2 there is substantiated the generalized polytropic model according to the Eddington method. The characteristics of the current Sun were calculated in Section 3 using the polytropic model with a spatially heterogeneous chemical composition. There were shown the advantages of such model compared with the standard model with $n = 3$. In the same section, there were calculated the evolutionary changes of the Sun's characteristics based on the modeling of the radial distribution of partial density of hydrogen. Section 4 is devoted to the influence of axial rotation on the characteristics of a star model, which is similar to the Sun, but has significant axial rotation.

2. The generalized equation of state

Using the Eddington method, we consider a model with spatially heterogeneous distribution of chemical composition, taking into account both gas and light pressure at the same time,

$$P_{\mathrm{gas}}(\mathbf{r}) = \frac{k_B}{m_u \mu(\mathbf{r})}\, \rho(\mathbf{r})\, T(\mathbf{r}), \quad P_{\mathrm{ph}}(\mathbf{r}) = \frac{a}{4}\, T^4(\mathbf{r}), \tag{1}$$

where $\rho(\mathbf{r})$ is the local density, $T(\mathbf{r})$ is the temperature, $\mu(\mathbf{r})$ is the local value of dimensionless (in atomic mass units m_u) molecular weight, $a = k_B^4(\hbar c)^{-3}\pi^2/15$, k_B is the Boltzmann constant, c is the speed of light. According to the main Eddington assumption

$$P_{\mathrm{gas}}(\mathbf{r}) = \beta P(\mathbf{r}), \quad P_{\mathrm{ph}}(\mathbf{r}) = (1-\beta)P(\mathbf{r}), \tag{2}$$

where $P(\mathbf{r})$ is the total pressure, and β is the constant independent on coordinates. By excluding the temperature from the system of equations (1) and (2), we obtain a relation between pressure and density in the form

$$P(\mathbf{r}) = K\left[\frac{\rho(\mathbf{r})}{f(\mathbf{r})}\right]^{4/3}, \quad f(\mathbf{r}) = \frac{\mu(\mathbf{r})}{\bar{\mu}}, \tag{3}$$

where $\bar{\mu}$ is the dimensionless parameter that represents the average value of the molecular weight by the volume of the star. Herewith,

$$K = \left\{\frac{1-\beta}{\beta^4} \cdot \frac{3}{a}\left(\frac{k_B}{m_u \bar{\mu}}\right)^4\right\}^{1/3} \tag{4}$$

coincides with the value of the constant in the Eddington model, namely in the approximation $\mu(\mathbf{r}) = \bar{\mu}$, or $f(\mathbf{r}) = 1$, that corresponds to the star of zero age on the main sequence. Using the example of the Sun, we will show that the equation of state (3) better describes the internal structure of the star compared to the standard polytropic model. This applies not only to the characteristics of the Sun in the current epoch, but also to their evolutionary changes.

3. The internal structure of the Sun within the generalized polytropic model

Taking into account a small angular velocity of the Sun ($\approx 3 \cdot 10^{-6}\mathrm{s}^{-1}$), we consider a spherically symmetrical model that corresponds to works performed on the system of Schwarzschild equations. We examine the mechanical equilibrium equation

$$\nabla P(r) = -\rho(r)\nabla\Phi_{\mathrm{grav}}(r), \tag{5}$$

in which

$$\Phi_{\rm grav}(r) = -G\int \frac{\rho(r^{'})d\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|} \tag{6}$$

is the gravitational potential on the sphere of radius r. Using expression (3) and taking into account that

$$\left[\frac{\rho(r)}{f(r)}\right]^{-2/3}\nabla\left(\frac{\rho(r)}{f(r)}\right) = 3\nabla\left(\frac{\rho(r)}{f(r)}\right)^{1/3}, \tag{7}$$

equation (5) takes the form

$$4K\nabla^2\left(\frac{\rho(r)}{f(r)}\right)^{1/3} = -f(r)\nabla^2\Phi_{\rm grav}(r) - (\nabla f(r), \nabla\Phi_{\rm grav}(r)). \tag{8}$$

In the spherically symmetrical model of a star with radius R

$$\Phi_{\rm grav}(r) = -\frac{GM(r)}{r} - 4\pi G\int\limits_r^R \rho(r')r'dr', \tag{9}$$

where

$$M(r) = 4\pi\int\limits_0^r \rho(r^{'})(r^{'})^2dr^{'} \tag{10}$$

is the mass of matter in the sphere of radius r, and $d\Phi_{\rm grav}(r)/dr = GM(r)/r^2$. According to the Poisson equation $\Delta\Phi_{\rm grav}(r) = 4\pi G\rho(r)$. Taking into account relation (9) and the next ones, we transform equation (8) to the integro-differential equation

$$4K\Delta\left(\frac{\rho(r)}{f(r)}\right)^{1/3} = -4\pi G\rho(r)f(r) - \frac{GM(r)}{r^2}\cdot\frac{df(r)}{dr}. \tag{11}$$

For a known function of $f(r)$, the solution of equation (11) determines the radial distribution of density $\rho(r)$. For a star of zero age $f(r) = 1$, equation (11) reduces to the differential equation for the standard polytropic model. For the convenience of integrating equation (11), let's introduce the dimensionless variables

$$\xi = r/\lambda, \quad y(\xi) = \left\{\frac{\rho(r)}{f(r)}\left[\frac{f_c}{\rho_c}\right]\right\}^{1/3}, \tag{12}$$

where λ is the scale of length, $\rho_c \equiv \rho(0)$, and $f_c \equiv f(0)$. Functions $\mu(r)$ and $f(r)$ are represented in the form

$$\begin{aligned} &\mu(r) \equiv \mu(r/R) = \mu(\xi/\xi_1), \quad f(r) \equiv f(r/R) \equiv \frac{\mu(\xi/\xi_1)}{\bar{\mu}} = \\ &= \frac{\mu(\xi/\xi_1)}{\mu(0)}\frac{\mu(0)}{\bar{\mu}} = f_c\frac{\mu(\xi/\xi_1)}{\mu(0)}, \\ &\frac{df(r)}{dr} = \frac{df(r/R)}{Rd(r/R)} = \frac{df(\xi/\xi_1)}{\lambda\xi_1 d(\xi/\xi_1)}, \end{aligned} \tag{13}$$

where $\xi_1 = R/\lambda$ is the dimensionless value of the star radius. Let's take into account that

$$\Delta = r^{-2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) = \frac{1}{\lambda^2}\Delta_\xi, \quad \Delta_\xi = \xi^{-2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\right),$$

let's multiply equation (11) by f_c/ρ_c and determine λ by

$$K = \pi G\left[\frac{\rho_c}{f_c}\right]^{2/3}\lambda^2. \tag{14}$$

As a result, we obtain the dimensionless form of the equilibrium equation

$$\Delta_\xi y(\xi) = -y^3(\xi)f^2(\xi/\xi_1) - \frac{f_c}{\xi_1}\cdot\frac{df(\xi/\xi_1)}{d(\xi/\xi_1)}\cdot\frac{1}{\xi^2}\int\limits_0^\xi(\xi')^2y^3(\xi')\,\frac{\mu(\xi'/\xi_1)}{\mu(0)}d\xi'. \tag{15}$$

According to the definition of the function $y(\xi)$, the regular solution of equation (15) must satisfy the boundary conditions

$$y(0) = 1, \quad \frac{dy}{d\xi} = 0 \ \text{ at } \ \xi = 0. \tag{16}$$

The dimensionless radius of the star ξ_1 is the root of equation $y(\xi) = 0$. Since ξ_1 plays the role of a parameter in equation (15), it is determined in a self-consistent way using the method of successive approximation during the numerical integration of this equation for a given function $f(\xi/\xi_1)$.

The function $y(\xi)$ and the dimensionless radius ξ_1 allow us to determine the unknown parameters of the problem λ, ρ_c and K from the system of equations

$$R_\odot = \lambda\,\xi_1, \quad M_\odot = 4\pi\lambda^3\rho_c\,\alpha, \quad K = \pi G\lambda^2\left(\frac{\rho_c}{f_c}\right)^{2/3} \tag{17}$$

with the known values of mass and radius of the Sun, where

$$\alpha = \int\limits_0^{\xi_1}\xi^2y^3(\xi)\frac{f(\xi/\xi_1)}{f(0)}d\xi = \int\limits_0^{\xi_1}\xi^2y^3(\xi)\frac{\mu(\xi/\xi_1)}{\mu(0)}d\xi. \tag{18}$$

3.1. The Sun characteristics within the standard polytropic model

The solution of the dimensionless equilibrium equation in the standard polytropic model ($f(r) = 1$) for $n = 3$ is well known and shown by curve 1 in Fig. 1. Herewith

$$\xi_1 = 6.89685\ldots, \qquad \alpha = 2.01824\ldots. \tag{19}$$

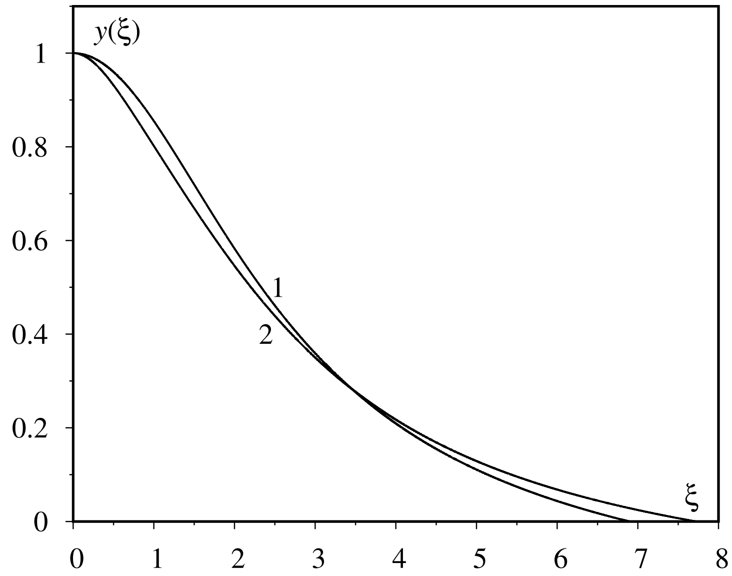

Figure 1. The solutions of the equilibrium equation. Curve 1 corresponds to the standard polytropic model, curve 2 – to the generalized model.

Using the observed values of the mass and radius of modern Sun ($M_\odot = 1.9891\ldots \cdot 10^{33}$g, $R_\odot = 6.9634\ldots \cdot 10^{10}$cm), we can determine the values of the parameters in the standard model

$$\begin{aligned} &\lambda = 1.0098 \cdot 10^{10}\ \text{cm}, \quad \rho_c = 76.1731\ \text{g}\,\text{cm}^{-3}, \\ &K = 3.8416 \cdot 10^{14}\ \text{cm}^3\,\text{g}^{-1/3}\,\text{s}^{-2}. \end{aligned} \tag{20}$$

The radial density distribution in such approximation is determined by the expression

$$\rho(r) = \rho_c y_3^3\left(\frac{r}{\lambda}\right) = \rho_c y_3^3(x\xi_1), \tag{21}$$

where $x \equiv r/R_\odot$, and $y_3(\xi)$ is Emden's function for the polytropic index $n = 3$. The obtained value of the central density is close to that found by Sears (1964) for the Sun of zero age through the numerical integration of the system of Schwarzschild equations ($\rho_c = 90\,\text{g}\,\text{cm}^{-3}$). If we use the value $R_\odot = 6.6460 \cdot 10^{10}$ cm (calculated by Sears (1964) for the zero age) instead of the modern radius of the Sun, then we obtain the specified values of the parameters

$$\begin{aligned} &\lambda = 0.9571 \cdot 10^{10}\ \text{cm}, \quad \rho_c = 87.6100\ \text{g}\,\text{cm}^{-3}, \\ &K = 3.8416 \cdot 10^{14}\ \text{cm}^3\,\text{g}^{-1/3}\,\text{s}^{-2}. \end{aligned} \tag{22}$$

Such value of the central density coincides with the value obtained by Sears (1964) for the Sun of zero age. This indicates that the standard model with $n = 3$ is entirely applicable to the Sun at zero age, where the spatial distribution of chemical elements is uniform and corresponds to the Eddington approximation. But such model is not applicable for the modern Sun, where the central density is close to the value $\rho_c = 158\ \mathrm{g\,cm^{-3}}$ (Sears, 1964).

3.2. The characteristics calculation in the generalized model of modern Sun

We use Emden's function for $n = 3$ as the zero approximation to find the solution of equation (15) by the iterative method. The coordinate dependence of the characteristics of the modern Sun was calculated through the numerical integration of the system of Schwarzschild equations by Sears (1964). Our aim is to compare the results of characteristic calculations for the modern Sun within the generalized model with those obtained by Sears (1964). For this, we use the coordinate dependence of the dimensionless molecular weight $\mu(r) \equiv \mu(r/R_\odot) = \mu(x)$, which was calculated by Lamers & Levesque (2017) for the values of the partial densities outside the core $X = 0.708$, $Y = 0.272$ and $Z = 0.020$. As shown in Fig. 2, the agreement is well-established almost everywhere in the interval $0 \leq x \leq 1$ (except for the surface layers) with the known expression

$$\mu(r/R_\odot) = \left\{ 2X(r/R_\odot) + \frac{3}{4}\, Y(r/R_\odot) + \frac{1}{2}\, Z(r/R_\odot) \right\}^{-1}, \tag{23}$$

which corresponds to the total ionization of matter. To simplify calculations, we do not take into account the change of $\mu(r/R_\odot)$ in the surface region, extrapolating instead the molecular weight value in the intermediate region to the surface region. We represent the function $\mu(x)$ in the form of a Padé approximant,

$$\begin{gathered} \mu(x) = \left\{ \sum_{j=0}^{3} b_j\, x^j \right\}^{-1} \sum_{i=0}^{3} a_i\, x^i, \\ a_0 = 0.0149173,\ a_1 = -0.0868327,\ a_2 = 0.730856,\ a_3 = 1.7342, \\ b_0 = 0.0172646,\ b_1 = -0.0893741,\ b_2 = 1.0339,\ b_3 = 2.96529. \end{gathered} \tag{24}$$

Calculating $\bar{\mu}$ as the average value of $\mu(r/R_\odot)$ over the star volume,

$$\bar{\mu} = 3R_\odot^{-3} \int_0^{R_\odot} \mu(r/R_\odot) r^2 dr = 3 \int_0^1 x^2 \mu(x) dx, \tag{25}$$

we represented in the analytical form the function $f(x) = \mu(x)/\bar{\mu}$, derivative df/dx, and $f(x) f_c^{-1} = \mu(x)/\mu(0)$. In Fig. 2 it is also shown the function $f(x)$ and

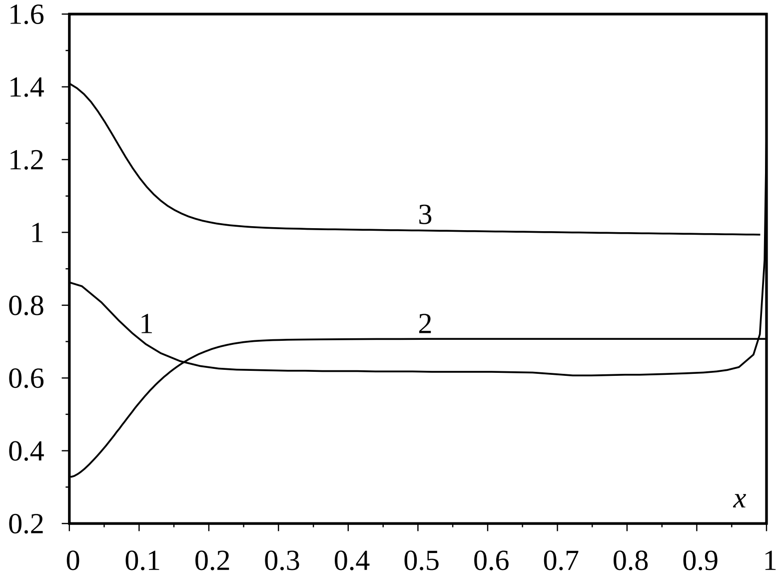

Figure 2. The coordinate dependence of the dimensionless molecular weight $\mu(x)$ (curve 1), the partial density of hydrogen $X(x)$ (curve 2) from Sears (1964) and Lamers & Levesque (2017), and the function $f(x)$ (curve 3).

the partial density of hydrogen $X(x)$ according to formula (23) and condition $X(x) + Y(x) + Z(x) = 1$. In order to analyze equation (15), we note that for a star with the age of the Sun, where thermonuclear reactions occur during $4.5{\cdot}10^9$ years, the molecular weight $\mu(r/R_\odot)$ in the core region is greater than the average value of $\bar{\mu}(f(r/R_0) > 1)$, and outside the core $\mu(r/R_\odot) \approx \bar{\mu}$, $(f(r) \approx 1)$. The nature of the solution to equation (15) is primarily determined by the first term on the right-hand side, while the second term (which is positive) plays the role of correction. Therefore, in the core region the condition $y(\xi) < y_3(\xi)$ must be satisfied, while outside the core the condition $y(\xi) > y_3(\xi)$ must be fulfilled. From there it follows that the dimensionless radius of the Sun ξ_1 in the generalized model must be greater than the value $\xi_1 = 6.89685\ldots$ of the standard model. The solution of equation (15) according to approximation (24) (curve 2) and the function $y_3(\xi)$ (curve 1) are shown in Fig. 1. In accordance with expressions (17) and (18) at $\xi_1 = 7.72441$ and $\alpha = 1.30993$, we have

$$\begin{aligned} &\lambda = 0.9015 \cdot 10^{10}\ \mathrm{cm}, \quad \rho_c = 164.9420\ \mathrm{g\,cm^{-3}}, \\ &K = 4.0776 \cdot 10^{14}\ \mathrm{cm^3\,g^{-1/3}\,s^{-2}}. \end{aligned} \tag{26}$$

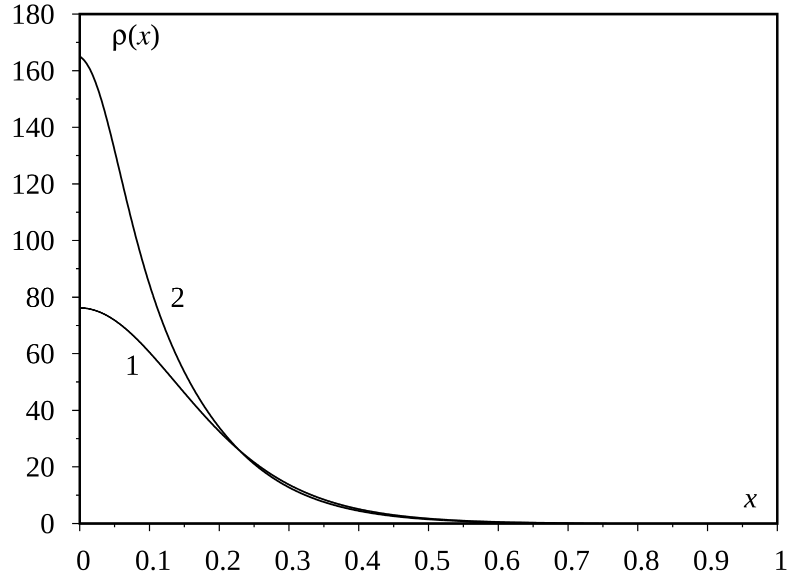

Figure 3. The distribution of density along the radius in different approximations. Curve 1 corresponds to the standard model, curve 2 – to the generalized model.

The density of matter is determined by the solution of equation (15),

$$\rho(r/R_\odot) = \rho(\xi/\xi_1) = \rho_c \frac{f(\xi/\xi_1)}{f_c} y^3(\xi). \tag{27}$$

In Fig. 3 the radial dependence of density compares in two models: curve 1 corresponds to the standard model, and curve 2 – to the generalized model. The radial dependence of density in the generalized model of the Sun, along with analogous values from Lamers & Levesque (2017), is shown in Fig. 4. The criterion for calculation is not only the central density but also other characteristics, including the gravitational and total energy of the star, moment of inertia, and the age of the star. To evaluate the age of the Sun, let's determine the hydrogen mass in the modern epoch

$$M_H = 4\pi\rho_c\lambda^3 \int_0^{\xi_1} \xi^2 y^3(\xi) X(\xi/\xi_1) \frac{\mu(\xi/\xi_1)}{\mu(0)} d\xi = M_\odot \frac{\gamma}{\alpha}, \tag{28}$$

where α is determined by formula (18), and

$$\gamma = \int_0^{\xi_1} \xi^2 y^3(\xi) X(\xi/\xi_1) \frac{\mu(\xi/\xi_1)}{\mu(0)} d\xi. \tag{29}$$

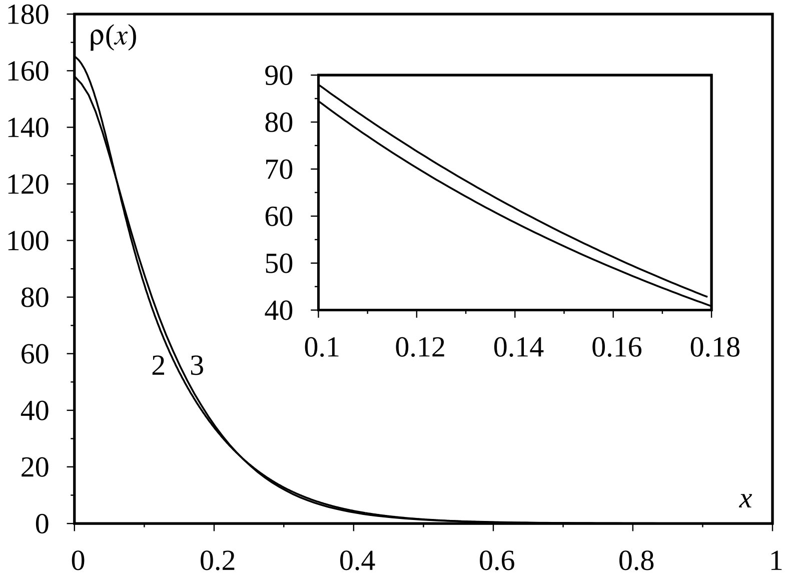

Figure 4. The distribution of density along the radius. Curve 2 corresponds to formula (15), curve 3 is taken from Lamers & Levesque (2017).

Since at the zero age phase $M_H = 0.708 \cdot M_\odot$, the loss of partial hydrogen mass is equal

$$\Delta M_H = \left(0.708 - \frac{\gamma}{\alpha}\right)\ M_\odot, \tag{30}$$

that corresponds to the mass defect $\delta_H = 0.00716 \cdot \Delta M_H$ and the radiation energy $\delta_H \cdot c^2$. Taking into account that the Sun's luminosity $L_\odot$ has changed only slightly during its existence, we can determine its age

$$t = \frac{\Delta M_H \cdot 0.00716 \cdot c^2}{L_\odot} = \frac{M_\odot}{L_\odot}\ \left(0.708 - \frac{\gamma}{\alpha}\right)\ 0.00716 \cdot c^2, \tag{31}$$

where $L_\odot$ is the Sun luminosity, c is the speed of light. For the modern Sun, $\gamma = 0.870$ and $\alpha = 1.310$, from which it follows that $t \approx 4.6 \cdot 10^9$ years. This value deviates from the generally accepted value by 2%. According to relations (17) and (27), the potential model energy is represented in the form

$$\begin{gathered} W = -\frac{1}{2} G \iint\limits_V \rho(r_1)\rho(r_2)|\mathbf{r}_1 - \mathbf{r}_2|^{-1} d\mathbf{r}_1 d\mathbf{r}_2 = -G\,\frac{M_\odot^2}{R_\odot} w, \\ w = \frac{\xi_1}{\alpha^2}\int\limits_0^{\xi_1} \xi\ y^3(\xi)\ \frac{\mu(\xi/\xi_1)}{\mu(0)} d\xi \int\limits_0^{\xi} \xi_2^2 y^3(\xi_2) \frac{\mu(\xi_2/\xi_1)}{\mu(0)} d\xi_2. \end{gathered} \tag{32}$$

For the modern Sun $w \cong 1.656$. Using the result of the calculation $\rho(r/R_\odot)$ from Lamers & Levesque (2017), we obtain the analogous expressions

$$\begin{aligned} W_s &= -G\,\frac{M_\odot^2}{R_\odot} w_s, \\ w_s &= \frac{1}{\alpha_s^2}\int_0^1 x\tilde{\rho}(x)dx \int_0^x z^2\tilde{\rho}(z)dz; \\ \alpha_s &= \int_0^1 x^2\,\tilde{\rho}(x)\,dx; \quad \tilde{\rho} = \rho(x)/\rho_c. \end{aligned} \tag{33}$$

As known from the polytropic theory, the volume density of internal energy

$$\mathcal{E}(\mathbf{r}) = 3P(\mathbf{r}) \tag{34}$$

for $n = 3$. In the case of a generalized polytrope

$$\mathcal{E}(\mathbf{r}) = 3K\left(\frac{\rho(\mathbf{r})}{f(\mathbf{r})}\right)^{4/3}, \tag{35}$$

therefore, the total internal energy

$$\begin{aligned} U &= \int_V \mathcal{E}(\mathbf{r})d\mathbf{r} = G\,\frac{M_\odot^2}{R_\odot}\,v, \\ v &= 0.75\,\xi_1\,(\alpha\,f_c)^{-2}\int_0^{\xi_1} \xi^2 y^4(\xi)\,d\xi \cong 1.655. \end{aligned} \tag{36}$$

As in the standard polytropic theory, in the generalized model gravitational energy and internal energy are mutually compensated. In connection with the redistribution of matter with age along the radius, it is worth comparing the moment of inertia relative to the Sun's diameter both in the standard and generalized models. In the generalized model

$$I = \int_V r^2\,\sin^2\theta\,\rho(r)\,d\mathbf{r} = M_\odot R_\odot^2\cdot 2\beta\,(3\alpha\,\xi_1^2)^{-1}, \tag{37}$$

where

$$\beta = \int_0^{\xi_1} \xi^4\,y^3(\xi)\,\frac{\mu(\xi/\xi_1)}{\mu(0)}\,d\xi. \tag{38}$$

For the modern Sun $I \cong 6.485\cdot 10^{53}$ g$\cdot$cm^2. To obtain the moment of inertia of the Sun of zero age, we should use $\alpha = 2.018$, and replace the multiplier $\mu(\xi/\xi_1)/\mu(0)$

with a unit. This yields the value $I(t=0) \cong 6.622 \cdot 10^{53}$ g $\cdot$ cm^2. Although the Sun's radius increases with the age, its moment of inertia decreases, leading to an increase in rotational velocity.

3.3. Evolutionary changes of the Sun characteristics

Comparing the characteristics of the Sun at zero age with its modern characteristics reveals an interesting problem in calculating their age dependence. For this purpose, we modeled the partial radial dependence of hydrogen in the Sun's core using curves that are shown in Fig. 5. The dotted straight line 1 corresponds to

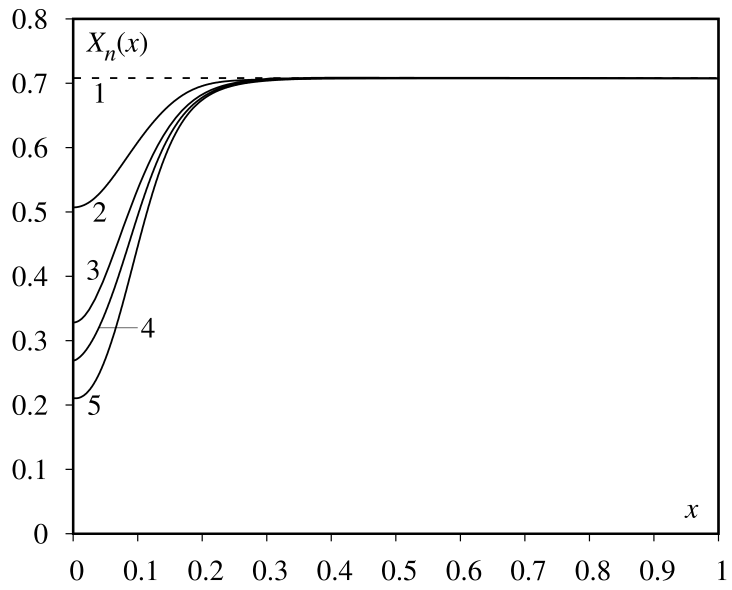

Figure 5. The partial distribution of hydrogen $X_n(x)$ in the Sun's core at different ages (see the text).

the zero age, curve 3 approximately corresponds to the current state of the Sun according to the calculations of Lamers & Levesque (2017). Curves 2, 4, and 5 are modeled to represent specific moments in the past and future. These curves are approximated using analytical expressions

$$X_n(x) = \left\{ \sum_{i=0}^{4} d_i^{(n)} x^i \right\}^{-1} \sum_{i=0}^{4} c_i^{(n)} x^i, \tag{39}$$

and the coefficients $c_i^{(n)}$, $d_j^{(n)}$ for $n = 2, 3, 4$, and 5 are represented in Tab. 1. Curves $X_n(x)$ correspond to the radial dependence of the dimensionless molec-

Table 1. The coefficients of formula (39).

n	$c_0^{(n)}$	$c_1^{(n)}$	$c_2^{(n)}$	$c_3^{(n)}$	$c_4^{(n)}$
2	0.00121312	−0.0129071	0.159383	−0.899003	2.39018
3	0.000923382	−0.0036938	0.177201	−0.69752	3.856
4	0.00018172	0.0010917	0.0298524	−0.0866747	2.41146
5	0.000393233	−0.00373364	0.0869501	−0.45848	2.55274
n	$d_0^{(n)}$	$d_1^{(n)}$	$d_2^{(n)}$	$d_3^{(n)}$	$d_4^{(n)}$
2	0.00239208	−0.0253873	0.252826	−1.31403	3.40033
3	0.00281341	−0.0112101	0.248745	−0.961501	5.43149
4	0.000676466	0.00341449	0.0237693	−0.0889951	3.38885
5	0.00186471	−0.0157796	0.159344	−0.706516	3.63906

ular weight $\mu_n(r/R_\odot)$, calculated according to expression

$$\mu_n(x) = \{0.75 + 1.25\ X_n(x) - 0.005\}^{-1}. \tag{40}$$

Herewith, $\mu_1 = \bar{\mu}_1 = 0.6135$, and $\mu_3(x)$ is represented by expression (24). For the models with functions $\mu_2(x)$, $\mu_4(x)$, and $\mu_5(x)$ the solutions of equation (15) were found, and the values of α_n, β_n and γ_n were calculated. Additionally, the central density $\rho_c^{(n)}$, the moment of inertia I_n were determined and the age of the model t_n was evaluated. All these values are shown in Tab. 2. The values of the Sun's radius for the corresponding age were calculated according to the approximation formula

$$R_\odot(t) = R_\odot(k_0 + k_1 t), \tag{41}$$

where t is expressed in billions of years, $R_\odot = 6.963 \cdot 10^{10}$ cm is the radius of modern Sun. The coefficients $k_0 = 0.9545$ and $k_1 = 0.0101$ are determined based on the known value of $R_\odot$ and the radius value for zero age $R_\odot(0) = 6.646 \cdot 10^{10}$ cm (Sears, 1964).

As it was shown in Tab. 2, during the evolution of the Sun on the main sequence, its radius increases, there is a significant redistribution of matter along the radius and, as a result, the moment of inertia decreases. The change of the Sun mass during the existence on the main sequence is not taken into account. Therefore, according to the law of conservation of angular momentum, the decreasing of moment of inertia causes the relative increasing of the angular rotation velocity by an order of magnitude $(1 \div 3)\%$ for 10^9 years depending on the age.

Table 2. Evolutionary changes of the Sun characteristics (here $R_{\odot}^{(n)}$ is determined in units 10^{10}cm, $\rho_c^{(n)}$ – g cm^{-3}, I_n – 10^{53}g cm^2, and t – 10^9 years).

n	$X(0)$	$\xi_1^{(n)}$	α_n	β_n	γ_n	$R_{\odot}^{(n)}$	$\rho_c^{(n)}$	I_n	t
1	0.708	6.896	2.01824	10.8516	1.42891	6.646	87.610	6.6219	0
2	0.507	7.336	1.63094	9.34026	1.12159	6.805	121.634	6.5370	2.1379
3	0.328	7.951	1.29411	8.08539	0.85976	6.963	182.081	6.3527	4.5937
4	0.269	8.407	1.16195	7.68088	0.75303	7.091	227.076	6.2378	6.3085
5	0.211	9.274	1.00958	7.35792	0.62507	7.305	320.827	5.9967	9.3548

4. The star structure with rotation within the generalized model

In this section we consider a model of a star with constant velocity of axial rotation $\omega = \mathrm{const}$, which in the absence of rotation would have the characteristics of the modern Sun. Our aim is to compare characteristics of two rotating polytropes – standard and generalized, in particular, changes in their characteristics under the influence of rotation. To simplify the problem, we will take into account the effect of rotation within the perturbation theory.

The equilibrium equation of the star with axial rotation generalizes relation (5),

$$\nabla P(\mathbf{r}) = -\rho(\mathbf{r}) \ \{\nabla \ \Phi_{\mathrm{grav}}(\mathbf{r}) + \nabla\Phi_{\mathrm{c}}(\mathbf{r})\} \,. \tag{42}$$

In the spherical coordinate system with the axis of rotation directed along the Oz direction, the centrifugal potential is

$$\Phi_{\mathrm{c}}(\mathbf{r}) = -\frac{1}{2}\,\omega^2 r^2 \sin^2\theta, \tag{43}$$

where θ is the polar angle. Using the equation of state (3) and applying the gradient operator to both sides of equation (42), we obtain the analogue of equation (8)

$$\begin{aligned} 4K\Delta \left(\frac{\rho(\mathbf{r})}{f(\mathbf{r})}\right)^{1/3} &= -4\pi G\rho(\mathbf{r})f(\mathbf{r}) - f(\mathbf{r})\Delta\Phi_{\mathrm{c}}(\mathbf{r}) - \\ &- \Big(\nabla f(\mathbf{r}), [\nabla\Phi_{\mathrm{grav}}(\mathbf{r}) + \nabla\Phi_{\mathrm{c}}(\mathbf{r})]\Big). \end{aligned} \tag{44}$$

At $f(\mathbf{r}) = 1$ this equations coincides with the equilibrium equation of a rotating polytrope. Since the last term on the right-hand side of equation (44) plays the role of correction, we consider it in the approximation of spherical symmetry, following the same way as in formula (8)

$$
\begin{aligned}
&4K\Delta\left(\frac{\rho(\mathbf{r})}{f(\mathbf{r})}\right)^{1/3} = -4\pi G\rho(\mathbf{r})f(\mathbf{r}) + 2\omega^2 f(\mathbf{r}) + \omega^2 r(1-z^2)\frac{\partial f(\mathbf{r})}{\partial r} - \\
&- \frac{GM(\mathbf{r})}{r^2}\cdot\frac{\partial f(\mathbf{r})}{\partial r},
\end{aligned} \tag{45}
$$

where $z = \cos\theta$. Due to the fact that the influence of rotation is taken into account as a perturbation, we use a spherically symmetrical approximation for the function $f(\mathbf{r})$. We perform transition to the dimensionless variables by relations

$$
r = \xi\lambda, \quad Y(\xi, z) = \left(\frac{\rho(\mathbf{r})}{f(\mathbf{r})}\cdot\frac{f_c}{\rho_c}\right)^{1/3}, \tag{46}
$$

using the same scale as for the model without rotation. Introducing the dimensionless angular velocity

$$
\Omega = \omega\left(\frac{f_c}{2\pi G\rho_c}\right)^{1/2}, \tag{47}
$$

we transform equation (45) to the dimensionless form

$$
\begin{aligned}
&\Delta_{\xi,z}Y(\xi, z) = \Omega^2 f(\xi/\xi_1) - Y^3(\xi, z) f^{\,2}(\xi/\xi_1) + \\
&+ \frac{\Omega^2}{2}\cdot\frac{\xi}{\xi_1}(1-z^2)\frac{\partial f(\xi/\xi_1)}{\partial(\xi/\xi_1)} - \frac{f_c}{\xi_1\xi^2}\cdot\frac{\partial f(\xi/\xi_1)}{\partial(\xi/\xi_1)}\int\limits_0^{\xi}(\xi^{'})^2 y^3(\xi^{'})\frac{\mu(\xi^{'}/\xi_1)}{\mu(0)}d\xi^{'}.
\end{aligned} \tag{48}
$$

Here

$$
\begin{aligned}
&\Delta_{\xi,z} = \Delta_\xi + \xi^{-2}\Delta_z; \\
&\Delta_\xi = \frac{\partial^2}{\partial\xi^2} + 2\xi^{-1}\frac{\partial}{\partial\xi}; \quad \Delta_z = \frac{\partial}{\partial z}(1-z^2)\frac{\partial}{\partial z}.
\end{aligned} \tag{49}
$$

The substitution

$$
Y(\xi, z) = y(\xi) + \Omega^2\Psi(\xi, z) \tag{50}
$$

and linearization of equation for the function $\Psi(\xi, z)$ predicts that the generalized polytrope, without rotation, is used in the role of a zero approximation for calculating characteristics of a rotating polytrope. In such approximation $\Psi(\xi, z)$ satisfies the equation

$$
\begin{aligned}
&\Delta\Psi(\xi, z) = f(\xi/\xi_1) + \xi(3\xi_1)^{-1}\Big(1 - P_2(z)\Big)\frac{\partial}{\partial(\xi/\xi_1)}f(\xi/\xi_1) - \\
&- 3y^2(\xi) f^{\,2}(\xi/\xi_1)\Psi(\xi, z),
\end{aligned} \tag{51}
$$

where $P_2(z)$ is the Legendre polynomial of second order and $y(\xi)$ is the solution of equation (15). From equation (51) it follows that $\Psi(\xi, z)$ can be represented in the form of expansion in a series of Legendre polynomials (Abramowitz & Stegun, 1970),

$$\Psi(\xi, z) = \psi_0(\xi) + \sum_{l\geq 1}^{\infty} a_{2l} P_{2l}(z)\psi_{2l}(\xi), \tag{52}$$

where a_{2l} are integration constants. By substituting series (52) into equation (51) and equating the multipliers with the same Legendre polynomials, we obtain the system of independent linear equations for functions $\psi_0(\xi)$ and $\psi_{2l}\xi$

$$\begin{aligned}
&\Delta_\xi \psi_0(\xi) = f(\xi/\xi_1) - 3y^2(\xi) f^2(\xi/\xi_1)\psi_0(\xi) + (3\xi_1)^{-1}\xi \frac{\partial}{\partial(\xi/\xi_1)} f(\xi/\xi_1);\\
&\Delta_\xi\ \psi_2(\xi) = \left\{\frac{6}{\xi^2} - 3y^2(\xi) f^2(\xi/\xi_1)\right\}\psi_2(\xi) -\\
&- (3a_2\xi_1)^{-1}\frac{\partial}{\partial(\xi/\xi_1)} f(\xi/\xi_1);\\
&\Delta_\xi \psi_{2l}(\xi) = \left\{\frac{2l(2l+1)}{\xi^2} - 3y^2(\xi) f^{\,2}(\xi/\xi_1)\right\}\psi_{2l}(\xi), \ldots
\end{aligned} \tag{53}$$

for $l \geq 2$. For $f(\xi/\xi_1) = 1$ these equations coincide with the equations for the standard rotational polytrope with $n = 3$ (Vavrukh et al., 2020). According to the definition of the function $Y(\xi, z)$ and conditions (16), equations (53) correspond to the boundary conditions

$$\psi_{2l}(0) = 0, \quad \frac{\partial}{\partial\xi}\ \psi_{2l}(\xi) = 0 \quad \text{at } \xi = 0 \ \text{ for } \quad l \geq 0. \tag{54}$$

It's easy to see that the function $\psi_0(\xi)$ has asymptotics $f(0)\xi^2/6 + \ldots$ for $\xi \ll 1$, and functions $\psi_{2l}(\xi)$ for $l \geq 1$ are convenient by normalized to the Bessel functions of the first kind (Abramowitz & Stegun, 1970) ($\psi_{2l}(\xi) \Rightarrow [(2l+1)!!]^{-1}\xi^{2l} + \ldots$). In Figs. 6 and 7 we compare the solutions of equations $\psi_0(\xi)$ and $\psi_2(\xi)$ for both the standard and the generalized rotational polytropes.

The expression

$$\rho_1(\xi, z) = \rho_c \frac{f(\xi/\xi_1)}{f_c}\left\{y(\xi) + \Omega^2\Psi(\xi, z)\right\}^3 \tag{55}$$

determines the distribution of matter in the rotational generalized polytrope, and the expression

$$\rho_0(\xi, z) = \rho_c^{(0)}\left\{y_3(\xi) + \Omega_0^2\Psi^{(0)}(\xi, z)\right\}^3 \tag{56}$$

yields the analogous distribution for the standard rotational polytrope. Herewith, $\Omega_0 = \omega(2\pi G\rho_c^{(0)})^{-1/2}$

$$\Psi^{(0)}(\xi, z) = \psi_0^{(0)}(\xi) + \sum_{l\geq 1} a_{2l}^{(0)}\psi_{2l}^{(0)}(\xi) P_{2l}(z), \tag{57}$$

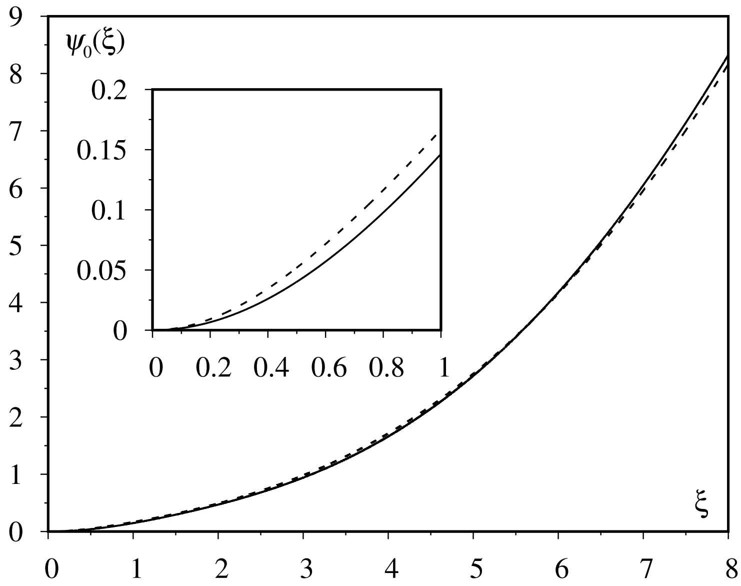

Figure 6. Dependence of the function $\psi_0(\xi)$ on the variable ξ. The solid curve corresponds to the standard model, the dashed one – to the generalized model.

and functions $\psi_0^{(0)}(\xi)$, $\psi_{2l}^{(0)}(\xi)$ are determined by equations (53), in which the replacements $f(\xi/\xi_1) \Rightarrow 1$ and $y(\xi) \Rightarrow y_3(\xi)$ should be performed.

In the case of small angular velocities, integration constants a_{2l} and $a_{2l}^{(0)}$ can be determined using the Milne-Chandrasekhar method (Milne, 1923) based on the condition of continuity of the gravitational potential on the star's surface. Constants a_2 and $a_2^{(0)}$ corresponding to the generalized and the standard models are determined by the following expressions

$$\begin{aligned} a_2 &= -\frac{5}{6}\xi_1^2 \left\{ 3\psi_2(\xi_1) + \xi_1 \frac{\partial}{\partial \xi_1}\psi_2(\xi_1) \right\}^{-1}, \\ a_2^{(0)} &= -\frac{5}{6}(\xi_1^{(0)})^2 \left\{ 3\psi_2^{(0)}(\xi_1^{(0)}) + \xi_1^{(0)} \frac{\partial}{\partial \xi_1^{(0)}}\psi_2^{(0)}(\xi_1^{(0)}) \right\}^{-1}, \end{aligned} \tag{58}$$

where $\xi_1^{(0)} = 6.89685\ldots$ is the dimensionless radius of the standard model without rotation. Note that in expressions (55) and (56), the central densities of corresponding polytropes without rotation appear.

The surface of rotational polytropes is determined in the following way. The conditions

$$Y(\xi, 1) = 0 \quad \text{and} \quad Y(\xi, 0) = 0 \tag{59}$$

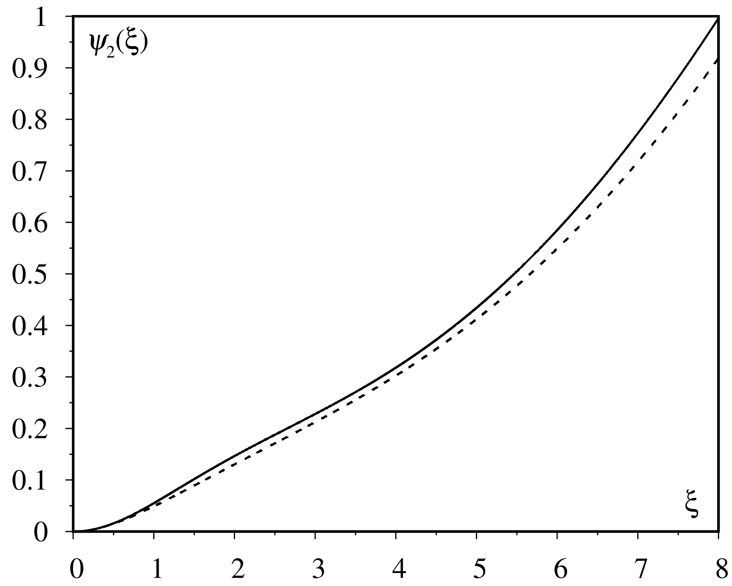

Figure 7. Dependence of the function $\psi_2(\xi)$ on the variable ξ. The notations are the same as in the previous figure.

determine polar and equatorial radii of the generalized rotational polytrope $\xi_p(\Omega)$ and $\xi_e(\Omega)$. Taking into account that the surface of the rotational polytrope is close to the surface of a rotational ellipsoid, the surface equation of the polytrope can be rewritten in the form

$$\xi_1(z) = \xi_e(\Omega) \left\{1 + z^2 \frac{e^2(\Omega)}{1 - e^2(\Omega)}\right\}^{-1/2}, \tag{60}$$

where

$$e(\Omega) = \left\{1 - \left(\frac{\xi_p(\Omega)}{\xi_e(\Omega)}\right)^2\right\}^{1/2} \tag{61}$$

is the eccentricity of the ellipsoid. In the case of the standard polytrope, we should make the replacement $\Omega \to \Omega_0$.

The dimensionless angular velocity Ω is an independent parameter of the problem. Its maximal value $\Omega_{\max}$ is determined by the conditions of the violation of monotonous behavior of the function $Y(\xi, z)$ at the star's equator

$$Y(\xi, 0) = 0, \qquad \frac{\partial}{\partial \xi}\, Y(\xi, 0) = 0, \tag{62}$$

which also determines the maximal value of the equatorial radius according to condition (59). For $\Omega > \Omega_{\max}$, there occurs a leakage of matter at the equator. For the standard model of a rotational polytrope with $n = 3$ we obtain the value that is close to $\Omega^{(0)}_{\max} = 0.0623$ (Vavrukh et al., 2020). In the case of the generalized polytrope, this value depends on the function $f(r/R)$. For the model that is close to the modern Sun, $\Omega_{\max} \approx 0.047\ldots$.

The values of polar and equatorial radii, as well as the eccentricity as functions of angular velocity in the interval $0 \leq \Omega \leq \Omega_{\max}$ are shown in Tab. 3.

Table 3. Dependence of the macroscopic characteristics of the generalized polytrope on the angular velocity Ω.

Ω	$\xi_p(\Omega)$	$\xi_e(\Omega)$	$e(\Omega)$	$\eta(\Omega)$
0.010	7.94171	8.0049	0.12540	1.00196
0.020	7.90259	8.16825	0.25296	1.00795
0.030	7.83899	8.50059	0.38678	1.01832
0.040	7.75314	9.21084	0.53988	1.03384
0.041	7.74343	9.33013	0.55786	1.03572
0.042	7.73353	9.46723	0.57682	1.03768
0.043	7.72344	9.62809	0.59701	1.03970
0.044	7.71317	9.82262	0.61918	1.0418
0.045	7.70271	10.0695	0.64408	1.04398
0.046	7.69207	10.4125	0.67400	1.04624
0.047	7.68126	11.0316	0.71775	1.04859

According to expression (55) the total mass of the generalized polytrope can be rewritten in the form

$$M(\Omega) = 2\pi\lambda^3\rho_c \int\limits_{-1}^{+1} dz \int\limits_{0}^{\xi_1(z)} \xi^2 \frac{f(\xi/\xi_1)}{f(0)} \{y(\xi) + \Omega^2\Psi(\xi, z)\}^3 d\xi. \qquad (63)$$

In Tab. 3 there is also shown the ratio

$$\eta(\Omega) = M(\Omega) \cdot M_3^{-1}, \qquad (64)$$

where M_3 is the mass of the generalized polytrope without rotation. In Fig. 8 there is shown the dependence of polar and equatorial radii for two rotational polytropes: the standard one with angular velocity Ω_0 and the generalized one with angular velocity Ω. The increase in the equatorial radius of the generalized polytrope under the influence of rotation is several times greater than the similar change for the standard polytrope.

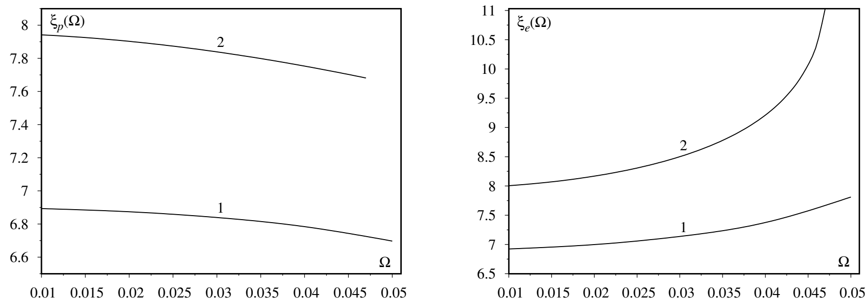

Figure 8. Dependence of polar and equatorial radii on the angular velocity in different approximations. Curves 1 correspond to the results of Vavrukh et al. (2020), curves 2 – to the generalized polytropic model.

5. Conclusions

1. The generalized polytropic model of stars with a spatially heterogeneous distribution of chemical composition, created by thermonuclear reactions, was substantiated using the Eddington method. This is manifested in the radial dependence of the dimensionless effective molecular weight. The standard polytropic model with $n = 3$ represents the limiting case when $f(r) = 1$ and corresponds to a star of zero age. This is confirmed through the calculation of characteristics of the Sun within both the standard and generalized models, and the comparison with the results of calculating the Sun's characteristics at zero age and in the modern epoch based on the Schwarzschild equations (Schwarzschild, 1958).

2. Modeling the radial distribution of partial density of hydrogen also allows us to calculate the evolutionary characteristics of the star, which are shown in Table 2. The table provides the central density, radius, and moment of inertia of the Sun at different epochs.

3. As shown in Section 4, the influence of rotation in the generalized polytropic model leads to much greater changes in mass, polar, and especially equatorial radii than in the case of the standard rotational polytrope. For example, for the maximal value of the dimensionless angular velocity ($\Omega_{\max} = 0.047\ldots$), the increase of mass and equatorial radius due to rotation is almost 3 times greater than in the standard rotational polytrope.

4. The calculations demonstrate the significant advantages of the generalized polytropic model. The results of Chandrasekhar (1933), James (1964), Caimmi (1980), Williams (1988), Kong et al. (2015), Knopik et al. (2017) and Vavrukh et al. (2020), which are based on the standard rotational polytropic model, correspond to stars with zero age and have a limited field of application.

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