REMARK ON THE SURFACES OF ZERO RELATIVE VELOCITY IN CLOSE BINARY SYSTEMS

I. Todoran
Astronomical Observatory, Cluj-Napoca, Romania

ABSTRACT. The purpose of the present note is to undertake a general survey on the surfaces of zero relative velocity and to discuss the hypothesis of axial rotation. So, in order to establish the corresponding equations, some consideration on axial rotation was necessary, at least as far as the two component stars are considered as mass-points.

The surfaces of zero relative velocity in close binary systems have proved their theoretical and practical importance. But, generally, such problems take into consideration the Roche model and, as it has been underlined by Plavec and Kratochvíl (1964), it is based on the following hypotheses:

1) The concentration of mass in the components is so high that their gravitational attraction is identical with that of mass-points.

2) The components revolve around the centre of gravity of the system in circular orbits.

3) Their axes of rotation are perpendicular to the orbital plane.

4) The periods of axial rotation are identical with the orbital revolution.

Now, having in mind the content of the first hypothesis, we want to prove that the last two above presented assumptions are superfluous at least as far as a qualitative study we have in view. In other words the surfaces of zero relative velocity in close binary systems may be studied without any consideration concerning the axial rotation of the components. Such a statement is supported by a particular solution of the restricted three-body problem.

In order to do so, let us first introduce here four rectangular coordinate systems. The first one (X_0, Y_0, Z_0) is at rest in an inertial frame of reference and its origin is taken as the common centre of mass, M, of the two stars S_1 and S_2. Here the direction of axes is chosen so that the plane MX_0Y_0 is comprised in the corresponding orbital plane. The second (X, Y, Z) has also its origin in M, but its positive X-axis will be directed toward the centre of the secondary component S_2.

The transformation equations between (X_o,Y_o,Z_o) and (X,Y,Z) are

$$X_o = X \cos\omega_k t - Y \sin\omega_k t$$
$$Y_o = X \sin\omega_k t + Y \cos\omega_k t \qquad (1)$$
$$Z_o = Z$$

where ω_k = const. represents the Keplerian angular velocity.

The third rectangular system (x,y,z) is at rest in a refence frame rotating with the orbital motion and has its origin at the centre of star S_1. Here we shall use the second hypothesis: The component stars revolve around the centre of gravity of the system in circular orbits. In such a consideration: $R_1=S_1M$, $R_2=MS_2$, $a=R_1+R_2$=const. and $X=x-R_1$, $Y=y$, $Z=z$ (see Fig. 1).

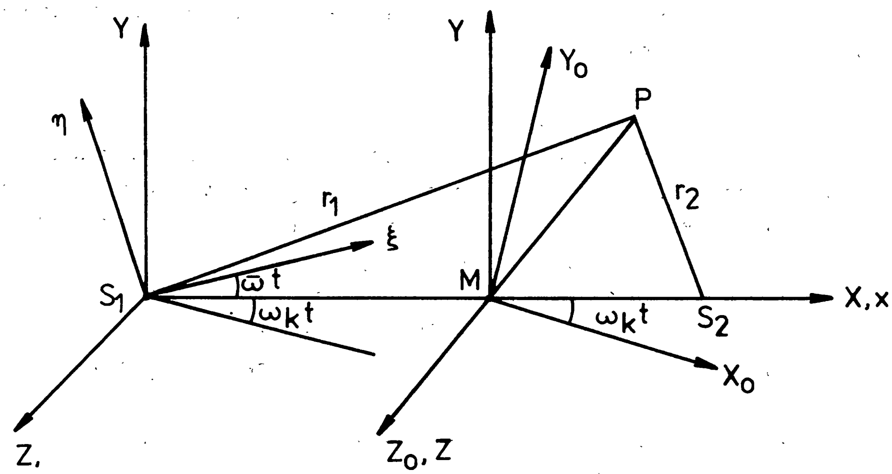

Fig. 1

The fourth rectangular coordinate system (ξ, η, ζ) has also its origin in the centre of star S_1, but it rotates with an arbitrary and constant angular velocity ϖ with respect to (x,y,z). The transformation equations between (x,y,z) and (ξ,η,ζ) are

$$x = \xi \cos\varpi t - \eta \sin\varpi t$$
$$y = \xi \sin\varpi t + \eta \cos\varpi t \qquad (2)$$
$$z = \zeta$$

and

$$r_1^2 = \xi^2 + \eta^2 + \zeta^2$$
$$r_2^2 = a^2 + \xi^2 + \eta^2 + \zeta^2 - 2a\,\xi \cos\varpi t + 2a\eta \sin\varpi t \qquad (3)$$
$$\omega = \varpi + \omega_k$$

In Fig. 1 P represents an infinitesimal body subjected to the attraction of the two stars S_1 and S_2, but it does not attract them. In such conditions the differential equations of motion for the considered infinitesimal body are,

$$\frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} = -\omega_k^2 R_1 \cos\varpi t + \frac{\partial\Omega}{\partial\xi}$$
$$\frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} = \omega_k^2 R_1 \sin\varpi t + \frac{\partial\Omega}{\partial\eta} \qquad (4)$$
$$\frac{d^2\zeta}{dt^2} = \frac{\partial\Omega}{\partial\zeta}$$

and

$$\Omega(\xi,\eta,\zeta) = \frac{\omega^2}{2}(\xi^2 + \eta^2) + G\frac{m_1}{r_1} + G\frac{m_2}{r_2} \qquad (5)$$

where G is gravitational constant, m_1 and m_2 are the masses of the two component stars. Here we have had in mind the first hypothesis:
The concentration of mass in components is so high that their gravitational attraction is identical with that of mass-points.

Multiplying Eqs. (4) respectively by $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$ and adding the resulting equations together, we have

$$\frac{2}{2}\left(\frac{d^2\xi}{dt^2}\frac{d\xi}{dt} + \frac{d^2\eta}{dt^2}\frac{d\eta}{dt} + \frac{d^2\zeta}{dt^2}\frac{d\zeta}{dt}\right) =$$

$$= -\omega_k^2 R_1\left(\frac{d\xi}{dt}\cos\varpi t - \frac{d\eta}{dt}\sin\varpi t\right) + \left(\frac{\partial\Omega}{\partial\xi}\frac{d\xi}{dt} + \frac{\partial\Omega}{\partial\eta}\frac{d\eta}{dt} + \frac{\partial\Omega}{\partial\zeta}\frac{d\zeta}{dt} + \frac{\partial\Omega}{\partial t}\right) -$$
$$- \frac{\partial\Omega}{\partial t} \qquad (6)$$

where $\frac{\partial\Omega}{\partial t}$ may be evaluated from (5) and (3). In doing so, we obtain

$$\frac{\partial \Omega}{\partial t} = - G\ m_2 \frac{a \varpi \xi \sin \varpi t}{r_2^3} - G\ m_2 \frac{a \varpi \eta \cos \varpi t}{r_2^3} = - \frac{G\ m_2 a \varpi}{r_2^3} (\xi \sin \varpi t + \eta \cos \varpi t). \quad (7)$$

In such conditions, after integration with respect to time, for V = 0, it is found that

$$G \frac{m_1}{r_1} + G \frac{m_2}{r_2} + \frac{(\varpi + \omega_k)^2}{2} (\xi^2 + \eta^2) + \int_{t_0}^{t} [(\omega_k^2 R_1 \frac{d\eta}{dt} + G\ m_2 \frac{a \varpi \xi}{r_2^3}) \sin \varpi t -$$

$$- (\omega_k^2 R_1 \frac{d\xi}{dt} - G\ m_2 \frac{a \varpi \eta}{r_2^3}) \cos \varpi t]\, dt = \text{const.} \quad (8)$$

which is an explicit function of time. Consequently, in such a case the surfaces of zero relative velocity are variable in time.

Here ϖ is an arbitrary angular velocity of the system (ξ, η, ζ) with respect to (x,y,z) and it has nothing in common with stellar axial rotation. Of course, in a peculiar case ϖ may be equal to stellar axial rotation, but in such a case some special problems have to be studied.

CONCLUDING REMARKS

1. It is easy to see that for $\varpi = 0$ the system (ξ, η, ζ) converts into the system (x,y,z) and Eq. (8) becomes the well known function

$$\Psi = G \frac{m_1}{r_1} + G \frac{m_2}{r_2} - \omega_k^2 R_1\ x + \frac{\omega_k^2}{2} (x^2 + y^2) = \text{const.}$$

2. If $\varpi = \omega_1$ = angular velocity of axial rotation of the star S_1, we are in the case studied by Huang (1967) where the new rotating system (ξ, η, ζ) is rigidly fixed in S_1 component.

3. Some of the above presented formulae have been used by different authors in different systems of unities (Plavec and Kratochvíl, 1964; Kruszewski, 1966; Huang, 1967 etc.), but it is very difficult to speak about the true effect of axial rotation, at least as far as the model of mass-points is accepted.

REFERNECES

Huang, S.S.: 1967, in Modern Astrophysics (A memorial to Otto Struve); Edited by M. Hack, Gauthier-Villars, Paris.

Kruszewski, A.: 1966, in Advances in Astronomy and Astrophysics 4, Edited by Z. Kopal. Acad. Press, New York and London.

Plavec, M., Kratochvíl, P.: 1964, Bull. Astron. Inst. Czechosl. 15,